

# On the Parallel Lines for Nondegenerate Conics

Rafał Abłamowicz<sup>1)</sup>

Jane Liu<sup>2)</sup>

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## Abstract

Computation of parallel lines (envelopes) to parabolas, ellipses, and hyperbolas is of importance in structure engineering and theory of mechanisms. Homogeneous polynomials that implicitly define parallel lines for the given offset to a conic are found by computing Gröbner bases for an elimination ideal of a suitably defined affine variety. Singularity of the lines is discussed and their singular points are explicitly found as functions of the offset and the parameters of the conic. Critical values of the offset are linked to the maximum curvature of each conic. Application to a finite element analysis is shown.

**Keywords:** Affine variety, elimination ideal, Gröbner basis, homogeneous polynomial, singularity, family of curves, envelope, pitch curve, undercutting, cam surface

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1) DEPARTMENT OF MATHEMATICS, BOX 5054, TENNESSEE TECHNOLOGICAL UNIVERSITY, COOKEVILLE, TN 38505, USA

E-MAIL: [rablamowicz@tntech.edu](mailto:rablamowicz@tntech.edu), URL: <http://math.tntech.edu/rafal/>

2) DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING, BOX 5015, TENNESSEE TECHNOLOGICAL UNIVERSITY, COOKEVILLE, TN 38505, USA

[jliu@tntech.edu](mailto:jliu@tntech.edu), [http://http://www.tntech.edu/cee/faculty/jane\\_liu.html](http://http://www.tntech.edu/cee/faculty/jane_liu.html)

<b>A</b>	<b>Appendix: Reduced Gröbner basis for the ideal</b>	$I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[y_0, x_0, x, y, r, p]$	<b>30</b>
<b>B</b>	<b>Appendix: Reduced Gröbner basis for the ideal</b>	$I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[y_0, x_0, x, y, a, b, r]$	<b>32</b>
<b>C</b>	<b>Appendix: Reduced Gröbner basis for the ideal</b>	$I_S = \langle h_1, h_2, h_3, h_4, h_5, g \rangle \subset \mathbb{R}[x, y, a, b, r]$	<b>37</b>

## 1 Introduction

The present paper introduces the Reader to a computation of parallel lines (envelopes) to three nondegenerate conics: a parabola, an ellipse, and a hyperbola. In each case, we show how to find a single polynomial  $g \in \mathbb{R}[x, y]$  that implicitly generates a suitable elimination ideal of an affine variety containing the parallel lines for the given offset  $r$  and for the given parameters of the conic. This amounts to a computation of a reduced Gröbner basis [4], [5], [7], [15] for the ideal for an appropriately chosen total order. We use **Maple** [9] and **Singular** [7] to accomplish computations. We show an application of the results presented here to a finite element analysis. We also make a connection with a design of surface cams in mechanical engineering [11], [13].

In each of the next three sections we discuss three cases of parallel lines for the specific values of the offset  $0 < r < r_{crit}$ ,  $r = r_{crit}$ , and  $r > r_{crit}$  for a parabola (Section 2), an ellipse (Section 3), and a hyperbola (Section 4). Then, in each section, we consider a general case of each conic while imposing essentially no condition on the offset except that  $r > 0$ . In Section 5 we show one possible application of the results to a finite element analysis of a composite material of elliptical shape. In Appendices A and B we collect some of the more cumbersome expressions while in Appendix C we show in detail how we found the singular points for the parallel lines of the ellipse.

## 2 Parallel Lines to a Parabola

In this section we compute parallel lines to a parabola defined by a polynomial  $f_1$  :

$$f_1 = 4py_0 - x_0^2 = 0 \quad (1)$$

where  $|p|$  denotes the distance between the focus  $F = (0, p)$  and the vertex  $V = (0, 0)$ .<sup>1</sup> Polynomial  $f_2$  defines a circle of radius (offset)  $r$  centered at a point  $(x_0, y_0)$  on the parabola  $f_1$ ,

$$f_2 = (y - y_0)^2 + (x - x_0)^2 - r^2 = 0 \quad (2)$$

while a polynomial  $f_3$

$$f_3 = 2xp - 2x_0p + x_0y - x_0y_0 = 0 \quad (3)$$

gives a condition that a point  $P(x, y)$  on the circle  $f_2$  lies on a line perpendicular to the parabola  $f_1$  and passing through the point  $(x_0, y_0)$  on the parabola. The family of such circles is called the *family of curves* determined by the polynomial  $f_1$  (see [4]). There are of course two such points for any given point  $(x_0, y_0)$  : one on each side of the parabola. All these points  $P$  belong to an affine variety  $\mathbf{V} = \mathbf{V}(f_1, f_2, f_3)$  – the *envelope* of the family of circles – and define two parallel lines at the distance  $r$  from the parabola. In what follows we will compute a reduced Gröbner basis for  $\mathbf{V}$  and by eliminating variables  $x_0, y_0$  we will obtain one polynomial  $g$  in variables  $x$  and  $y$  alone (also  $p, r$  in a general case) that defines both parallel lines on each side of the parabola for the given value of the offset  $r$ . Thus, in general, we have  $f_1, f_2, f_3 \in \mathbb{R}[x_0, y_0, x, y, p, r]$ . However, first we illustrate computation for specific values of the parameters  $p$  and  $r$ , and later we show computation for unspecified parameters.

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<sup>1</sup>Without any loss of generality, we assume that the parabola has its vertex at the origin and its focus on the  $y$ -axis.

## 2.1 Parallel lines for specific parameters $p$ and $r$

The parallel lines constitute a subvariety of the affine variety  $\mathbf{V} = \mathbf{V}(I)$  where  $I = \langle f_1, f_2, f_3 \rangle$ . In order to find the polynomial  $g$  for the second elimination ideal  $I_2 = I \cap \mathbb{R}[x, y]$ , we first find a Gröbner basis  $G$  for  $\mathbf{V}$  for the lexicographic order  $y_0 > x_0 > x > y$  that contains sixteen polynomials.<sup>2</sup> Then, we reduce  $G$  to a minimal Gröbner basis  $G_m$  that contains six (or eight, depending on the case) polynomials and, further, to a unique reduced Gröbner basis  $G_r$  with three (or five, depending on the case) polynomials only. Finally, we eliminate from  $G_r$  all polynomials that contain variables  $y_0$  and  $x_0$ , and keep the single polynomial  $g \in \mathbb{R}[x, y]$  that gives the parallel lines and generates the second elimination ideal  $I_2 = \langle g \rangle$ .

We are also interested when the parallel lines form a smooth (non singular) variety  $\mathbf{V}(g)$  or a  $C^\infty$  1-manifold or a *hypersurface defined by  $g$* . Since  $g$  can be viewed as a  $C^\infty$  function  $\mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\mathbf{V}(g) = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\}$ , it is known that  $\mathbf{V}(g)$  is a smooth manifold at those points  $p \in \mathbf{V}(g)$  where  $(\nabla g)(p) \neq 0$ . [10, Theorem 2.6] We will find singular points of  $\mathbf{V}(g)$  by finding points  $p \in \mathbf{V}(g)$  where  $(\nabla g)(p) = 0$ .

Thus,  $\mathbf{V}(g)$  can be thought of as a union of points where the gradient is zero and those points where the gradient is not zero. Connected components of  $\mathbf{V}(g)$  where the gradient is not zero form the 1-dimensional manifold of the parallel lines. However, we are also interested in describing a discrete finite set  $S$  of singular points where the gradient is zero. For a formal definition of a singular point of an affine variety  $\mathbf{V}(g)$  see [4, page 136].

**Example 1** ( $r < r_{crit}$ ) Let  $p = \frac{1}{3}$  and  $r = \frac{1}{4}$ . Then,

$$\mathbf{V} = \mathbf{V}(4y_0 - 3x_0^2, 16y^2 - 32yy_0 + 16y_0^2 + 16x^2 - 32xx_0 + 16x_0^2 - 1, 2x - 2x_0 + 3x_0y - 3x_0y_0)$$

We find that the reduced Gröbner basis  $G_r$  contains only three polynomials including

$$\begin{aligned} g = & 331776x^6 - 42192x^2 + 48400y^2 - 448512y^3 - 84480x^2y + 589824y^4 + 28032y \\ & - 25344x^4 + 1138176x^2y^2 - 884736x^2y^3 - 1105920x^4y - 5329 + 331776x^4y^2 \end{aligned} \quad (4)$$

whose gradient is

$$\begin{aligned} \nabla g = & (-879x - 1760yx - 1056x^3 + 23712xy^2 - 46080x^3y - 18432y^3x + 13824x^3y^2 + 20736x^5)\mathbf{i} \\ & + (3025y - 2640x^2 - 42048y^2 + 876 + 71136x^2y - 34560x^4 - 82944x^2y^2 + 73728y^3 + 20736x^4y)\mathbf{j}. \end{aligned} \quad (5)$$

Thus, the only real point  $S_1$  on  $\mathbf{V}(g)$  where  $(\nabla g)(S_1) = 0$  is  $S_1 = (0, \frac{73}{192})$ . We will refer to this point as a virtual singular point for the following reason:  $\mathbf{V}(g)$  is a disjoint union of  $\{S_1\}$  and a non-singular smooth subvariety  $\Pi$  where the gradient  $\nabla g$  does not vanish. The subvariety  $\Pi$  constitutes the set of two parallel lines that we have been seeking. We show all in Figure 1.

We have verified by direct computation that as the value of the offset  $r$  increases towards some critical value  $r_{crit}$ , the virtual singular point approaches the smooth component of the variety  $\mathbf{V}(g)$ , that is, it approaches the parallel lines  $\Pi$ .<sup>3</sup> As the next example shows, when  $r = r_{crit}$ , then the virtual singular point is actually located on the smooth component  $\Pi$  of the variety  $\mathbf{V}(g)$ .

<sup>2</sup>Computations have been performed with **Groebner** package in **Maple** 8 [9] as well as with a small package written by the Authors [2], and the results have been confirmed with **Singular** [7] using [1].

<sup>3</sup>In the next subsection we will derive a formula for the critical value  $r_{crit}$  of  $r$  for a general parabola.

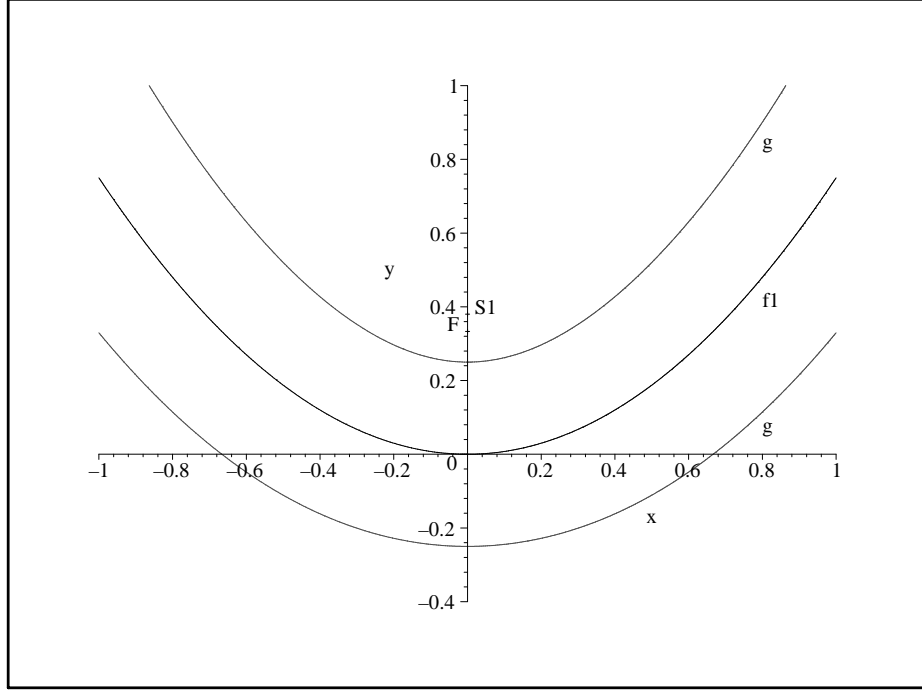


Figure 1: Parabola with parallel lines  $g$ , focus  $F$  and one virtual singular point  $S_1$  for  $p = \frac{1}{3}$  and  $r = \frac{1}{4}$

**Example 2** ( $r = r_{crit}$ ) Let  $p = \frac{1}{3}$  and  $r = \frac{2}{3}$ . Then,

$$\mathbf{V} = \mathbf{V}(4y_0 - 3x_0^2, 9y^2 - 18yy_0 + 9y_0^2 + 9x^2 - 18xx_0 + 9x_0^2 - 4, 2x - 2x_0 + 3x_0y - 3x_0y_0)$$

In this case, we have found that the reduced Gröbner basis  $G_r$  contains five polynomials including

$$g = 768y - 288x^2 - 1728y^3 + 1944x^2y^2 - 891x^4 - 2430x^4y - 1944x^2y^3 + 1296y^4 + 729x^4y^2 + 729x^6 - 256 \quad (6)$$

whose gradient is

$$\begin{aligned} \nabla g = & (-32x + 216xy^2 - 198x^3 - 540x^3y - 216y^3x + 162x^3y^2 + 243x^5)\mathbf{i} \\ & + (128 - 864y^2 + 648x^2y - 405x^4 - 972x^2y^2 + 864y^3 + 243x^4y)\mathbf{j}. \end{aligned} \quad (7)$$

Thus, like in Example 1, there is only one real point  $S_1$  on  $\mathbf{V}(g)$  where  $(\nabla g)(S_1) = 0$  : it is  $S_1 = (0, \frac{2}{3})$ . However, unlike in the first example, this time point  $S_1$  is not disjoint from the rest of the variety  $\mathbf{V}(g)$  as it can be seen in Figure 2. We will refer to this point as a singular point of the variety  $\mathbf{V}(g)$ . The latter can still be thought of as union of  $\{S_1\}$  and the non-singular smooth subvariety  $\Pi$  where the gradient  $\nabla g$  does not vanish. We will say that in this case the parallel lines possess one singular point.

Probably the most interesting case occurs when the value of the offset  $r > r_{crit}$  as shown in the next example. In this case, the single singular point of the variety  $\mathbf{V}(g)$  splits into three singular points.

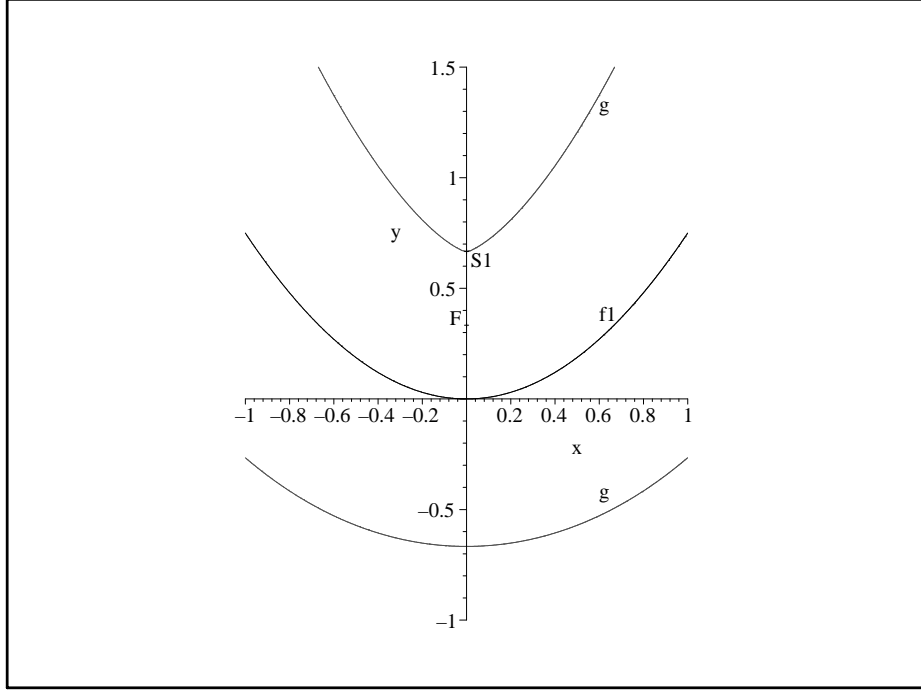


Figure 2: Parabola with parallel lines  $g$ , focus  $F$  and one singular point  $S_1$  for  $p = \frac{1}{3}$  and  $r = \frac{2}{3}$

**Example 3** ( $r > r_{crit}$ ) Let  $p = \frac{1}{3}$  and  $r = \frac{3}{2}$ . Then,

$$\mathbf{V} = \mathbf{V}(4y_0 - 3x_0^2, 4y^2 - 8yy_0 + 4y_0^2 + 4x^2 - 8xx_0 + 4x_0^2 - 9, 2x - 2x_0 + 3x_0y - 3x_0y_0)$$

In this case, we have found that the reduced Gröbner basis  $G_r$  again contains only three polynomials including

$$g = 83808y + 52812x^2 + 16900y^2 - 37248y^3 - 4896x^2y^2 - 34416x^4 - 17280x^4y - 13824x^2y^3 + 9216y^4 + 5184x^4y^2 + 5184x^6 - 84681 + 6240x^2y \quad (8)$$

whose gradient is

$$\nabla g = (4401x - 408xy^2 - 5736x^3 - 2880x^3y - 1152y^3x + 864x^3y^2 + 1296x^5 + 520yx)\mathbf{i} + (10476 + 4225y - 13968y^2 - 1224x^2y - 2160x^4 - 5184x^2y^2 + 4608y^3 + 1296x^4y + 780x^2)\mathbf{j}. \quad (9)$$

Thus, unlike in the above two examples, this time there are three singular points:  $S_1 = (0, \frac{97}{48})$  located on the  $y$ -axis (the symmetry axis of the parabola), and two additional points  $S_2$  and  $S_3$

$$S_2 = \left( \frac{\sqrt{65 + 36 \cdot 12^{\frac{1}{3}} - 27 \cdot 12^{\frac{2}{3}}}}{6}, \frac{3 \cdot 12^{\frac{1}{3}}}{4} - \frac{1}{3} \right), \quad (10)$$

$$S_3 = \left( -\frac{\sqrt{65 + 36 \cdot 12^{\frac{1}{3}} - 27 \cdot 12^{\frac{2}{3}}}}{6}, \frac{3 \cdot 12^{\frac{1}{3}}}{4} - \frac{1}{3} \right) \quad (11)$$

located symmetrically on each side of the  $y$ -axis as shown in Figure 3. We will say that in this case the parallel lines possess three singular points. A similar figure can be found in [4, page 141]

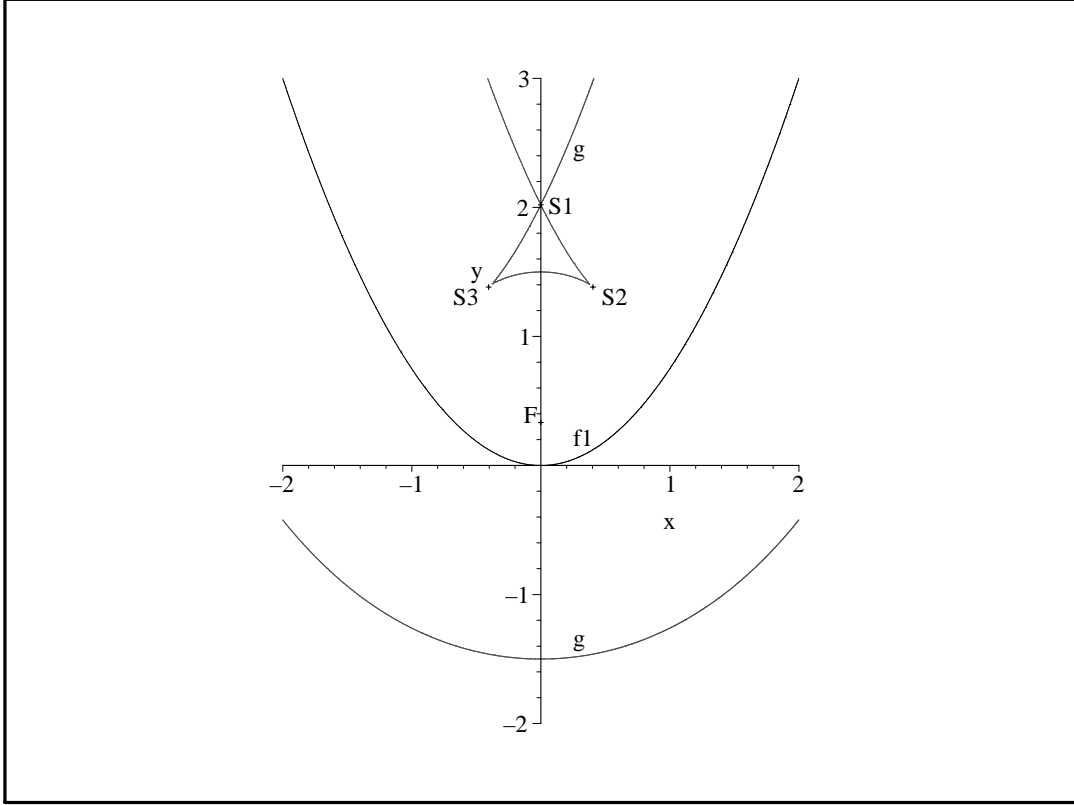


Figure 3: Parabola with parallel lines  $g$ , focus  $F$  and three singular points  $S_1, S_2, S_3$  for  $p = \frac{1}{3}$  and  $r = \frac{3}{2}$

## 2.2 Parallel lines for arbitrary parameters $p$ and $r$

In this section we will find a general polynomial  $g$  that will generate the second elimination ideal  $I_2 = I \cap \mathbb{R}[x, y, p, r]$  where  $I = \langle f_1, f_2, f_3 \rangle$  and the generating polynomials are defined in equations (1), (2), and (3). We will also find a formula for the critical value  $r_{crit}$  of  $r$  that determines whether  $\mathbf{V}(I_2)$  has one or three singular points, and, we will find general formulas for the coordinates of the said singular points for arbitrary values of the parameters  $p$  and  $r$ .

We begin with a computation of a Gröbner basis for the ideal  $I$  for the lexicographic order  $y_0 > x_0 > x > y > r > p$ .

$$I = \langle 4py_0 - x_0^2, y^2 - 2yy_0 + y_0^2 + x^2 - 2xx_0 + x_0^2 - r^2, 2xp - 2x_0p + x_0y - x_0y_0 \rangle \subset \mathbb{R}[y_0, x_0, x, y, r, p]$$

Using **Maple** and **Singular** [1], [7] we found that a reduced Gröbner basis for  $I$  consists of fourteen homogeneous polynomials of degrees: 7 (two polynomials), 6 (two polynomials), 5 (two polynomials), 4 (two polynomials), 3 (three polynomials), and 2 (three polynomials). These polynomials are displayed in Appendix A.

Thus, in this general case we observe that  $I_2 = \langle g \rangle$  where  $g = g_1/p$  is the following homogeneous polynomial in  $\mathbb{R}[x, y, r, p]$  of degree six:

$$\begin{aligned} g = & -2pr^2yx^2 + 8pr^2y^3 + 8p^2r^2y^2 - 32yp^3r^2 + 16p^4r^2 - 16y^4p^2 + 32y^3p^3 \\ & - 16p^4y^2 + 3r^2x^4 + 8p^2r^4 + 20p^2r^2x^2 - y^2x^4 + 10ypx^4 - x^6 - x^4p^2 + 8py^3x^2 \\ & - 32x^2y^2p^2 + 8x^2yp^3 - 3r^4x^2 + 2r^2x^2y^2 + r^6 - r^4y^2 - 8pr^4y \quad (12) \end{aligned}$$

We want to find now the singular points on the variety  $\mathbf{V} = \mathbf{V}(g)$  where the gradient  $\nabla g$  is zero. Thus,  $\mathbf{V}$  can be thought of as union of points where the gradient is zero and those points where the gradient is not zero. Connected components of  $\mathbf{V}$  where the gradient is not zero constitute the 1-dimensional manifold that we encountered above. We are interested in the finite set  $S$  of the singular points of the variety. We will show next that the critical value of  $r$  is  $r_{crit} = |2p| = 2|p|$ . Therefore, we compute  $\nabla g = \nabla g(x, y, r, p)$  :

$$\begin{aligned} \nabla g = & < -x(2ypr^2 - 6x^2r^2 - 20p^2r^2 + 2x^2y^2 - 20ypx^2 + 3x^4 + 2x^2p^2 - 8py^3 + 32y^2p^2 - 8yp^3 + 3r^4 - 2r^2y^2), \\ & - x^2pr^2 + 12y^2pr^2 + 8yp^2r^2 - 16p^3r^2 - 32y^3p^2 + 48p^3y^2 - 16yp^4 - yx^4 + 5x^4p + 12x^2py^2 - 32x^2p^2y \\ & + 4x^2p^3 + 2r^2yx^2 - r^4y - 4pr^4, \\ & r(-2ypx^2 + 8py^3 + 8y^2p^2 - 32yp^3 + 16p^4 + 3x^4 + 16p^2r^2 + 20x^2p^2 - 6x^2r^2 + 2x^2y^2 + 3r^4 \\ & - 2r^2y^2 - 16ypr^2), \\ & - r^2yx^2 + 4r^2y^3 + 8y^2pr^2 - 48yp^2r^2 + 32p^3r^2 - 16y^4p + 48y^3p^2 - 32p^3y^2 + 8pr^4 + 20x^2p^2r^2 \\ & + 5yx^4 - x^4p + 4y^3x^2 - 32x^2py^2 + 12x^2p^2y - 4r^4y > \quad (13) \end{aligned}$$

In order to find solutions to the vector equation  $\nabla g = 0$ , we need to solve in fact a resulting system of four polynomial equations in variables  $x, y, p, r$ . The system could be simplified by assuming  $r > 0$ , yet to efficiently analyze its solutions one can again employ the Gröbner basis approach and find a reduced Gröbner basis for the ideal  $I_S$  generated by the four equations. Since we want to make sure that the solutions belong to  $\mathbf{V}(g)$ , we add equation  $g = 0$  to our system. Let  $F$  be a list consisting of the five polynomial equations, that is,  $F = [\nabla g = 0, g = 0]$  and let  $I_S = \langle \nabla g, g \rangle$ . Thus, we find that the affine variety  $\mathbf{V}(I_S)$  of singular points is generated by a reduced Gröbner basis consisting of eleven homogeneous polynomials  $s_i, i = 1, \dots, 11$ , in  $y, x, r$ , and  $p$  with total degrees varying from 5 through 17. The third polynomial  $s_3$  in the basis has the following form:

$$\begin{aligned} s_3 = & xr(640p^6y - 320p^7 + 3r^4yx^2 - 3r^2yx^4 - 12pr^2x^4 + 48x^2p^3r^2 - r^6y + 14r^6p \\ & + 84p^3r^4 - 480p^5r^2 + yx^6 + 13x^6p - 24x^4p^3 - 384x^2p^5 - 96yp^2r^2x^2 - 15pr^4x^2 \\ & - 60yp^2r^4 + 96yp^4r^2 - 60yx^4p^2 - 96x^2p^4y) \quad (14) \end{aligned}$$

It appears that since  $r$  and  $p$  are non-zero, one possible solution to the equation  $s_3 = 0$  is  $x = 0$ . Let's then see if the remaining polynomials in  $F$  can be solved when  $x = 0$ . Upon substituting  $x = 0$  into the list  $F$ , one obtains

$$\begin{aligned} [0, & -(-4py + r^2 + 4p^2)(-8py^2 + yr^2 + 4pr^2 + 4yp^2), r(-4py + r^2 + 4p^2)(-2y^2 + 3r^2 - 4py + 4p^2), \\ & 4(-y + 2p)(r - y)(r + y)(-4py + r^2 + 4p^2), (r - y)(r + y)(-4py + r^2 + 4p^2)] \quad (15) \end{aligned}$$

Since  $r > 0$ , one possible solution to (15) is  $r = y$  which gives the following system:

$$F_1 = [0, -(r-2p)^4 r, (r-2p)^4 r, 0, 0] \quad (16)$$

This leads to the critical value of  $r_{crit} = 2p$ . Of course, this makes sense only when  $p > 0$ , that is, in general we have  $r_{crit} = |2p| = 2|p|$ . When  $p < 0$ , we can still get the same critical value of  $r$  when we select  $r = -y$  in (15) which leads to

$$F_2 = [0, (r+2p)^4 r, (r+2p)^4 r, 0, 0]. \quad (17)$$

Thus, the critical value of  $r$  in this case is  $r_{crit} = -2p$ . Combining both cases we get  $r_{crit} = |2p| = 2|p|$ .

Another solution of the system (15) can be found by setting  $y = \frac{4p^2 + r^2}{4p} = p + \frac{r^2}{4p}$  ( $p \neq 0$ ). This gives the singular point

$$S_1 = \left(0, \frac{r^2 + 4p^2}{4p}\right) = \left(0, p + \frac{r^2}{4p}\right) \quad \text{when } p \neq 0 \quad \text{and } r > 0 \quad (18)$$

found in Examples 1, 2 and 3 above. When  $0 < r < r_{crit} = 2|p|$ , then this is the only singular point  $S_1$  as shown in Figure 1 that is not located on the continuous part of  $\mathbf{V}(g)$ : It is located at the distance  $\frac{r^2}{4|p|}$  above the focus (when  $p > 0$ ) and below the focus (when  $p < 0$ ) on the symmetry line of the parabola (the  $y$ -axis in our case). Then, the parallel line is smooth at the points  $(0, r)$  and  $(0, -r)$ . These points are located on the  $y$  axis above and below the vertex of the parabola. As  $r$  approaches  $r_{crit}$  from the left, then  $S_1$  approaches one of these points: It approaches the point located on the parallel line on the concave side of the parabola (see Figure 1). When  $r = r_{crit}$  then the singular point  $S_1$  is located on the parallel line on the concave side of the parabola (see Figure 2).

Now we look for those singular points on  $\mathbf{V}(g)$  for which  $x \neq 0$ . These points will be present only when  $r > r_{crit}$ . Thus, we now proceed to solve the system of five polynomial equations contained in the list  $F$ , or, equivalently, the system of eleven polynomial equations  $s_i = 0, i = 1, \dots, 11$ , in the Gröbner basis for  $I_S$  under the condition that  $x \neq 0$ . Using approach described in more details in Appendix C for the ellipse, in the case when  $r > r_{crit}$  one obtains the following two additional singular points

$$S_2 = \left(\alpha, \frac{\beta}{\gamma}\right) \quad \text{and} \quad S_3 = \left(-\alpha, \frac{\beta}{\gamma}\right) \quad \text{when } p \neq 0 \quad \text{and } r > r_{crit} = 2|p|. \quad (19)$$

The expressions  $\alpha, \beta$  and  $\gamma$  are as follows:

$$\alpha = \frac{\sqrt{(pr^2)^{\frac{2}{3}}r^2 + 6r^2p^22^{\frac{1}{3}} - 3(pr^2)^{\frac{1}{3}}pr^22^{\frac{2}{3}} - 4(pr^2)^{\frac{2}{3}}p^2}}{\sqrt[3]{pr^2}}, \quad (20)$$

$$\begin{aligned} \beta = & pr^2(22r^6(pr^2)^{\frac{2}{3}} + 1452r^6p^22^{\frac{1}{3}} + 7456p^3r^42^{\frac{2}{3}}(pr^2)^{\frac{1}{3}} - 6560p^2r^4(pr^2)^{\frac{2}{3}} \\ & - 15488p^4r^42^{\frac{1}{3}} - 39680p^5r^22^{\frac{2}{3}}(pr^2)^{\frac{1}{3}} + 37600p^4r^2(pr^2)^{\frac{2}{3}} + 7936p^82^{\frac{1}{3}} \\ & - 15872p^72^{\frac{2}{3}}(pr^2)^{\frac{1}{3}} + 45760p^6r^22^{\frac{1}{3}} - 33r^82^{\frac{1}{3}} - 5376p^6(pr^2)^{\frac{2}{3}}), \end{aligned} \quad (21)$$

$$\begin{aligned} \gamma = & 2(pr^2)^{\frac{1}{3}}(8640p^6r^2(pr^2)^{\frac{1}{3}} - 3968p^72^{\frac{1}{3}}(pr^2)^{\frac{2}{3}} + 1984p^7r^22^{\frac{2}{3}} \\ & - 3920r^4p^4(pr^2)^{\frac{1}{3}} - 9920p^5r^22^{\frac{1}{3}}(pr^2)^{\frac{2}{3}} + 4960p^5r^42^{\frac{2}{3}} + 484r^6p^2(pr^2)^{\frac{1}{3}} \\ & + 1864p^3r^42^{\frac{1}{3}}(pr^2)^{\frac{2}{3}} - 932p^3r^62^{\frac{2}{3}} - 11r^8(pr^2)^{\frac{1}{3}}). \end{aligned} \quad (22)$$

Thus, in summary, we have these three cases:



- (i) One (virtual) singular point  $S_1 = (0, p + \frac{r^2}{4p})$  when  $r < r_{crit} = 2|p|$ . In particular, when  $p = \frac{1}{3}$  and  $r = \frac{1}{4}$ , we get from (18) that  $S_1 = (0, \frac{73}{192})$ : That is the only (discrete) singular point of  $\mathbf{V}(g)$  found in Example 1. Formula (12) then gives

$$g = \frac{5329}{331776} - \frac{3025}{20736}y^2 - \frac{73}{864}y + \frac{55}{216}yx^2 + \frac{73}{54}y^3 + \frac{293}{2304}x^2 \\ + \frac{11}{144}x^4 - \frac{247}{72}x^2y^2 - \frac{16}{9}y^4 + \frac{10}{3}yx^4 + \frac{8}{3}y^3x^2 - x^6 - y^2x^4$$

which is, up to the form and sign, the same as polynomial (4).

- (ii) One (true) singular point  $S_1 = (0, p + \frac{r^2}{4p})$  when  $r = r_{crit} = 2|p|$ . In particular, when  $p = \frac{1}{3}$  and  $r = \frac{2}{3}$ , we get from (18) that  $S_1 = (0, \frac{2}{3})$ : That is the only singular point of located on the continuous part of  $\mathbf{V}(g)$  found in Example 2. Formula (12) then gives

$$g = \frac{256}{729} - \frac{256}{243}y + \frac{64}{27}y^3 + \frac{32}{81}x^2 + \frac{11}{9}x^4 - \frac{8}{3}x^2y^2 - \frac{16}{9}y^4 + \frac{10}{3}yx^4 + \frac{8}{3}y^3x^2 - x^6 - y^2x^4$$

which is, up to the form and sign, the same as polynomial (6).

- (iii) Three singular points:  $S_1 = (0, p + \frac{r^2}{4p})$ ,  $S_2 = (\alpha, \frac{\beta}{\gamma})$ , and  $S_3 = (-\alpha, \frac{\beta}{\gamma})$  when  $r > r_{crit} = 2|p|$ . In particular, when  $p = \frac{1}{3}$  and  $r = \frac{3}{2}$ , we get from (18) that  $S_1 = (0, \frac{97}{48})$  while from (19) we get the two additional singular points  $S_2$  and  $S_3$  displayed in (10) and (11) found in Example 3. Formula (12) then gives

$$g = \frac{9409}{576} - \frac{4225}{1296}y^2 - \frac{97}{6}y - \frac{65}{54}yx^2 + \frac{194}{27}y^3 - \frac{163}{16}x^2 + \frac{239}{36}x^4 \\ + \frac{17}{18}x^2y^2 - \frac{16}{9}y^4 + \frac{10}{3}yx^4 + \frac{8}{3}y^3x^2 - x^6 - y^2x^4$$

which is, up to the form and sign, the same as polynomial (8).

As a final comment, let's notice that for the parabola  $4p(y - k) = (x - h)^2$  the critical value of the offset

$$r_{crit} = 2|p| = \frac{1}{\kappa_{max}} = \rho_{min}$$

is the reciprocal of  $\kappa_{max}$ , the maximum curvature of the parabola at its vertex, or, equivalently, it equals  $\rho_{min}$ , the minimum radius of the osculating circle.<sup>4</sup> Furthermore, the quantity  $2|p|$  is known as the *semi-latus rectum* and represents the distance from the parabola focus to the parabola measured along a line parallel to the parabola directrix. [14]

### 3 Parallel Lines to an Ellipse

In this section we compute various parallel lines to an ellipse defined by a polynomial

$$f_1 = b^2y_0^2 + a^2x_0^2 - a^2b^2 = 0 \tag{23}$$

---

<sup>4</sup>Recall that when  $\beta(t)$  is a regular curve in  $\mathbb{R}^3$  then  $\kappa = |\dot{\beta} \times \ddot{\beta}|/|\dot{\beta}|^3$ . [10, page 46].

where we assume that  $a > b$  and the center of the ellipse is at the origin. Polynomial  $f_2$  defines a circle of radius (offset)  $r$  centered at a point  $P = (x_0, y_0)$  on the ellipse,

$$f_2 = (y - y_0)^2 + (x - x_0)^2 - r^2 = 0 \quad (24)$$

while polynomial  $f_3$

$$f_3 = b^2(x - x_0)y_0 - a^2(y - y_0)x_0 = 0 \quad (25)$$

gives a condition that a point  $P(x, y)$  on the circle  $f_2$  lies on a line perpendicular to the ellipse  $f_1$  and passing through the point  $(x_0, y_0)$  on the ellipse. All these points  $P$  belong to the affine variety  $\mathbf{V} = \mathbf{V}(f_1, f_2, f_3)$  – the envelope – and, like for the parabola, define two parallel lines at the distance  $r$  from the ellipse. We proceed like in Section 2: We will compute a reduced Gröbner basis for  $\mathbf{V}$  and by eliminating variables  $x_0, y_0$  we will obtain one polynomial  $g$  in variables  $x$  and  $y$  alone (also  $a, b, r$  in a general case) that defines both parallel lines on each side of the ellipse for the given value of the offset  $r$ . Thus, in general, we have  $f_1, f_2, f_3 \in \mathbb{R}[y_0, x_0, x, y, a, b, r]$ . However, first we illustrate computations for specific values of the parameters  $a, b$  and  $r$ , and later we show computation for unspecified parameters.

### 3.1 Parallel lines for specific parameters $a, b, r$

The parallel lines constitute a subvariety of the affine variety  $\mathbf{V} = \mathbf{V}(I)$  where, like before,  $I = \langle f_1, f_2, f_3 \rangle$ . In order to find the polynomial  $g$  for the second elimination ideal  $I_2 = I \cap \mathbb{R}[x, y]$ , we first find a Gröbner basis  $G$  for  $\mathbf{V}$  for the lexicographic order  $y_0 > x_0 > x > y$  and reduce it eventually to a unique reduced basis  $G_r$  of nine (or eleven, depending on the case) polynomials only. Finally, we eliminate from that basis all polynomials that contain variables  $y_0$  and  $x_0$ , and keep the single polynomial  $g \in \mathbb{R}[x, y]$  that gives the parallel lines and generates the second elimination ideal  $I_2 = \langle g \rangle$ . We will also proceed to find all singular points  $p$  of  $\mathbf{V}(g)$  where  $(\nabla g)(p) = 0$ . Thus, let's consider three cases when  $0 < r < r_{crit}$ ,  $r = r_{crit}$ , and  $r > r_{crit}$ . In the next section we will consider a general case for arbitrary values of  $a, b$  ( $a > b > 0$ ) and  $r$  ( $r > 0$ ).

**Example 4** ( $r < r_{crit}$ ) Let  $a = 3$ ,  $b = \frac{3}{2}$ , and  $r = \frac{1}{2}$ . Then,

$$\mathbf{V} = \mathbf{V}(y_0^2 + 4x_0^2 - 9, 4y^2 - 8yy_0 + 4y_0^2 + 4x^2 - 8xx_0 + 4x_0^2 - 1, y_0x + 3y_0x_0 - 4x_0y)$$

We find that the reduced Gröbner basis  $G_r$  contains nine polynomials including

$$g = 256x^8 + 2080x^6 - 41685x^2 + 44100 - 25356y^2 + 5353y^4 - 4751x^4 - 488y^6 - 3360y^4x^2 - 4680y^2x^4 + 528y^4x^4 + 640x^6y^2 + 160y^6x^2 + 16y^8 + 21410x^2y^2 \quad (26)$$

whose gradient is

$$\begin{aligned} \nabla g = & x(1024x^6 + 6240x^4 - 41685 - 9502x^2 - 3360y^4 - 9360x^2y^2 + 1056y^4x^2 \\ & + 1920y^2x^4 + 160y^6 + 21410y^2)\mathbf{i} \\ & + y(-12678 + 5353y^2 - 732y^4 - 3360x^2y^2 - 2340x^4 + 528y^2x^4 + 320x^6 \\ & + 240y^4x^2 + 32y^6 + 10705x^2)\mathbf{j}. \end{aligned} \quad (27)$$

There are exactly two solutions to the equation  $(\nabla g)(p) = 0$  on  $\mathbf{V}(g)$ :

$$S_1 = (0, \sqrt{6}) \quad \text{and} \quad S_2 = (0, -\sqrt{6})$$

We will refer to these points as virtual singular points for the following reason:  $\mathbf{V}(g)$  is a disjoint union of  $\{S_1, S_2\}$  and a non-singular smooth subvariety  $\Pi$  where the gradient  $\nabla g$  does not vanish. The subvariety  $\Pi$  constitutes the set of two parallel lines that we have been seeking. Note that the foci  $F_1 = (0, \frac{3\sqrt{3}}{2})$  and  $F_2 = (0, -\frac{3\sqrt{3}}{2})$  are different from the singular points. We show all in Figure 4.

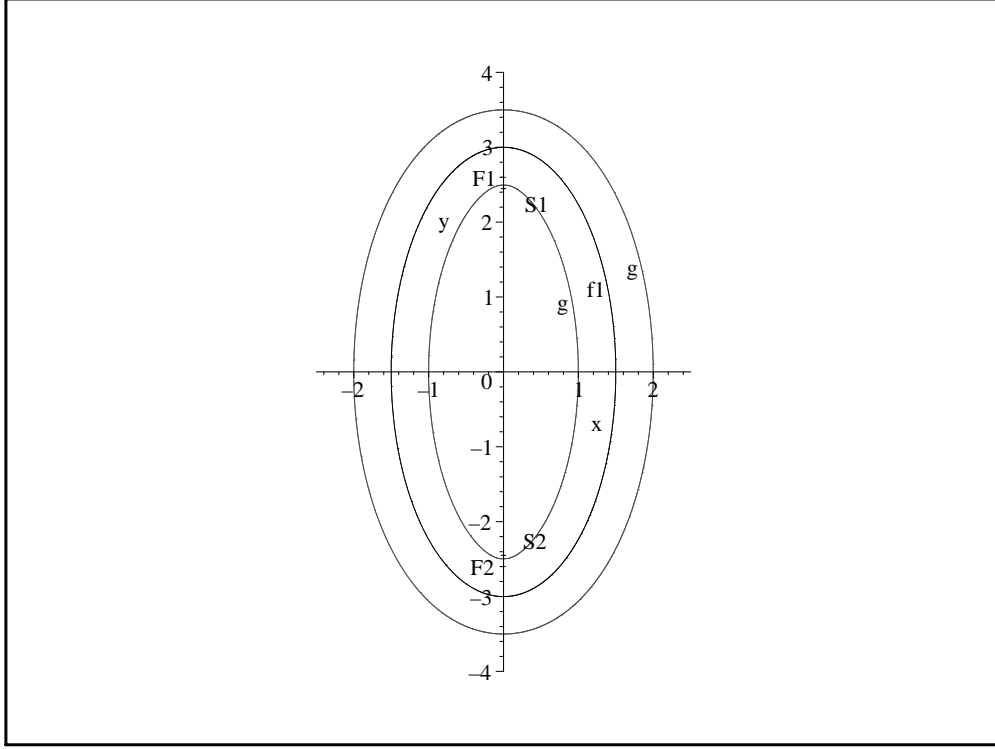


Figure 4: Ellipse with parallel lines  $g$ , foci  $F_1, F_2$  and two singular points  $S_1, S_2$  for  $a = 3, b = \frac{3}{2}$  and  $r = \frac{1}{2}$

We have verified by direct computation that as the value of the offset  $r$  increases towards some critical value  $r_{crit}$ , the virtual singular points approach the smooth component of the variety  $\mathbf{V}(g)$ , that is, they approach the parallel lines  $\Pi$ .<sup>5</sup> As the next example shows, when  $r = r_{crit}$ , then the virtual singular points are actually located on the smooth component  $\Pi$  of the variety  $\mathbf{V}(g)$ .

**Example 5** ( $r = r_{crit}$ ) Let  $a = 3, b = \frac{3}{2}$ , and  $r = \frac{3}{4}$ . Then,

$$\mathbf{V} = \mathbf{V}(y_0^2 + 4x_0^2 - 9, 16y^2 - 32yy_0 + 16y_0^2 + 16x^2 - 32xx_0 + 16x_0^2 - 9, y_0x + 3y_0x_0 - 4x_0y)$$

We find that the reduced Gröbner basis  $G_r$  contains eleven polynomials including

$$\begin{aligned} g = & -29673216x^4 + 19035648y^4 + 95178240x^2y^2 - 1916928y^6 - 14376960y^4x^2 \\ & - 21012480y^2x^4 + 7372800x^6 + 65536y^8 + 2162688y^4x^4 + 655360y^6x^2 \\ & + 2621440x^6y^2 + 1048576x^8 - 79361856y^2 + 119574225 - 198404640x^2 \end{aligned} \quad (28)$$

<sup>5</sup>In the next subsection we will derive a formula for the critical value  $r_{crit}$  of  $r$  for a general ellipse.

whose gradient is

$$\begin{aligned}\nabla g = & x(-1854576x^2 + 2974320y^2 - 449280y^4 - 1313280x^2y^2 + 691200x^4 + 135168y^4x^2 \\ & + 20480y^6 + 245760y^2x^4 + 131072x^6 - 6200145)\mathbf{i} \\ & + y(594864y^2 + 1487160x^2 - 89856y^4 - 449280x^2y^2 - 328320x^4 + 4096y^6 \\ & + 67584y^2x^4 + 30720y^4x^2 + 40960x^6 - 1240029)\mathbf{j}.\end{aligned}\quad (29)$$

There are exactly two solutions to the equation  $(\nabla g)(p) = 0$  on  $\mathbf{V}(g)$  :

$$S_1 = \left(0, \frac{9}{4}\right) \quad \text{and} \quad S_2 = \left(0, -\frac{9}{4}\right)$$

These points are the singular points of the variety  $\mathbf{V}(g)$  : At every other point of this variety the gradient  $\nabla g$  does not vanish. We will collect these points as before into a non-singular subvariety  $\Pi$ . The variety  $\Pi$  constitutes the set of two parallel lines that we have been seeking. Note that the foci  $F_1 = (0, \frac{3\sqrt{3}}{2})$  and  $F_2 = (0, -\frac{3\sqrt{3}}{2})$  are still different from the singular points. We show all in Figure 5.

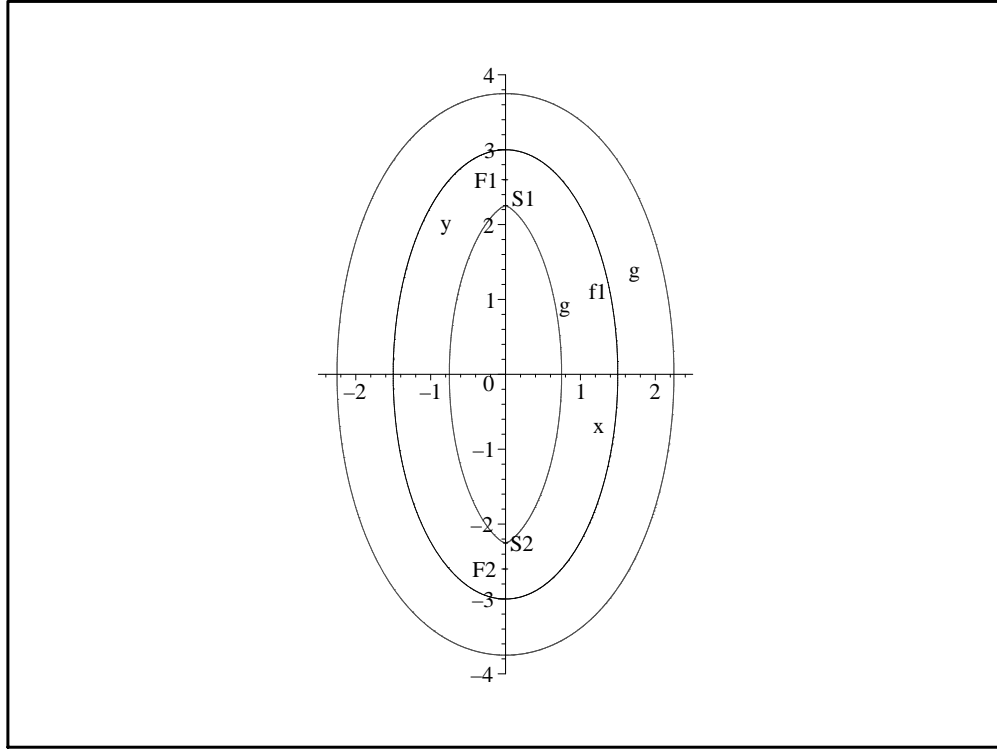


Figure 5: Ellipse with parallel lines  $g$ , foci  $F_1, F_2$  and two singular points  $S_1, S_2$  for  $a = 3$ ,  $b = \frac{3}{2}$  and  $r = \frac{3}{4}$

As we can see from the last figure, when  $r = r_{crit} = \frac{3}{4}$  for the given values of  $a$  and  $b$ , the singular points are located on the continuous component of  $\Pi$ .

Probably the most interesting parallel lines we obtain when  $r > r_{crit}$  as we show in the next example. There, we will find six singular points as each of the two singular points found in Example 5 splits into three points.

**Example 6** ( $r > r_{crit}$ ) Let  $a = 3$ ,  $b = \frac{3}{2}$ , and  $r = \frac{3}{2}$ . Then,

$$\mathbf{V} = \mathbf{V}(y_0^2 + 4x_0^2 - 9, 9y^2 - 18yy_0 + 9y_0^2 + 9x^2 - 18xx_0 + 9x_0^2 - 16, y_0x + 3y_0x_0 - 4x_0y)$$

We find that again the reduced Gröbner basis  $G_r$  contains nine polynomials including

$$\begin{aligned} g = & 23461560x^2y^2 + 1221025 + 186624x^8 - 2984040y^4x^2 - 5015520y^2x^4 \\ & - 284472y^6 + 384912y^4x^4 - 43658160x^2 + 518400x^6 + 466560x^6y^2 \\ & + 116640y^6x^2 + 11664y^8 - 2228394y^2 - 10769184x^4 + 1344177y^4 \end{aligned} \quad (30)$$

whose gradient is

$$\begin{aligned} \nabla g = & x(977565y^2 + 31104x^6 - 124335y^4 - 417960x^2y^2 + 32076y^4x^2 - 1819090 + 64800x^4 \\ & + 58320y^2x^4 + 4860y^6 - 897432x^2)\mathbf{i} \\ & + y(3910260x^2 - 994680x^2y^2 - 835920x^4 - 142236y^4 + 128304y^2x^4 + 77760x^6 \\ & + 58320y^4x^2 + 7776y^6 - 371399 + 448059y^2)\mathbf{j}. \end{aligned} \quad (31)$$

There are exactly six solutions to the equation  $(\nabla g)(p) = 0$  on  $\mathbf{V}(g)$ , or, in another words, six singular points of the variety:

$$\begin{aligned} S_1 &= \left(0, \frac{\sqrt{51}}{6}\right) \quad \text{and} \quad S_2 = \left(0, -\frac{\sqrt{51}}{6}\right) \\ S_3 &= \left(\frac{\sqrt{525 + 324 \cdot 6^{\frac{2}{3}} - 864 \cdot 6^{\frac{1}{3}}}}{18}, \frac{2\sqrt{231 - 81 \cdot 6^{\frac{2}{3}} + 54 \cdot 6^{\frac{1}{3}}}}{9}\right), \\ S_4 &= \left(-\frac{\sqrt{525 + 324 \cdot 6^{\frac{2}{3}} - 864 \cdot 6^{\frac{1}{3}}}}{18}, \frac{2\sqrt{231 - 81 \cdot 6^{\frac{2}{3}} + 54 \cdot 6^{\frac{1}{3}}}}{9}\right), \\ S_5 &= \left(\frac{\sqrt{525 + 324 \cdot 6^{\frac{2}{3}} - 864 \cdot 6^{\frac{1}{3}}}}{18}, -\frac{2\sqrt{231 - 81 \cdot 6^{\frac{2}{3}} + 54 \cdot 6^{\frac{1}{3}}}}{9}\right), \\ S_6 &= \left(-\frac{\sqrt{525 + 324 \cdot 6^{\frac{2}{3}} - 864 \cdot 6^{\frac{1}{3}}}}{18}, -\frac{2\sqrt{231 - 81 \cdot 6^{\frac{2}{3}} + 54 \cdot 6^{\frac{1}{3}}}}{9}\right). \end{aligned}$$

We display these six points in Figure 6.

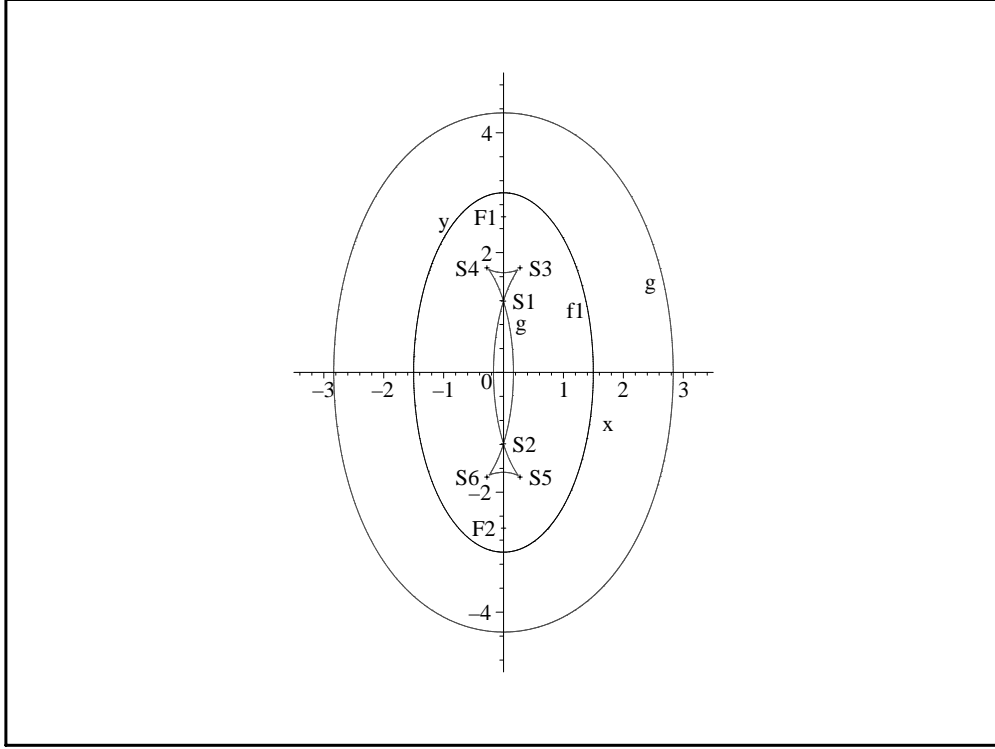


Figure 6: Ellipse with parallel lines  $g$ , foci  $F_1, F_2$  and six singular points  $S_i, i = 1, \dots, 6$ , for  $a = 3$ ,  $b = \frac{3}{2}$  and  $r = \frac{4}{3}$

### 3.2 Parallel lines for arbitrary parameters $a, b, r$

In this section we will find a general polynomial  $g$  that will generate the second elimination ideal  $I_2 = I \cap \mathbb{R}[x, y, a, b, r]$  where  $I = \langle f_1, f_2, f_3 \rangle$  and the generating polynomials are defined in equations (23), (24), and (25). We will also find a formula for the critical value  $r_{crit}$  of  $r$  that determines whether  $\mathbf{V}(I_2)$  has two or six singular points, and, we will find general formulas for the coordinates of the said singular points for arbitrary values of the parameters  $a, b$  and  $r$ .

We begin with a computation of a Gröbner basis for the ideal  $I$  for the lexicographic order  $y_0 > x_0 > x > y > a > b > r$ .

$$I = \langle b^2 y_0^2 + a^2 x_0^2 - a^2 b^2, y^2 - 2yy_0 + y_0^2 + x^2 - 2xx_0 + x_0^2 - r^2, b^2 y_0 x - b^2 y_0 x_0 - a^2 x_0 y + a^2 x_0 y_0 \rangle$$

We found that a reduced Gröbner basis for  $I$  consists of fifteen homogeneous polynomials of degrees: 16 (two polynomials), 14 (one polynomial), 12 (one polynomial), 11 (two polynomials), 10 (two polynomials), 9 (one polynomial), 8 (one polynomial), 7 (two polynomials), 5 (one polynomial), 4 (one polynomial), and 2 (one polynomial). These polynomials are displayed in Appendix B.

Thus, in this general case we observe that  $I_2 = \langle g \rangle$  where  $g = g_1/(a^2 b^2)$  is the following homogeneous

polynomial in  $\mathbb{R}[x, y, a, b, r]$  of degree twelve<sup>6</sup>:

$$\begin{aligned}
g = & 6b^4x^4a^4 - 4y^6b^4a^2 - 6a^6x^4b^2 - 4b^6x^2a^4 + 6b^4x^2a^6 + a^4b^8 - 2y^2b^8a^2 \\
& + 6y^2b^6x^2a^2 + 6a^6y^2x^2b^2 - 10b^4x^2a^4y^2 + y^8b^4 + b^8r^4 + b^4r^8 - 2b^6r^6 \\
& - 2b^4y^2x^4r^2 - 6x^2y^4b^4r^2 + 6x^2b^4r^4y^2 - 4b^4y^6r^2 + 2b^4y^6x^2 + y^4b^4x^4 \\
& + b^4x^4r^4 - 2b^4r^6x^2 - 2b^6x^2r^4 + 6y^4b^4r^4 - 2b^8y^2r^2 + 6b^6r^4y^2 - 4b^4r^6y^2 \\
& + 6a^4y^2b^6 + 6a^6y^2b^2r^2 + 2a^4b^2x^4y^2 - 4a^4x^6b^2 + 4a^6x^2b^2r^2 - 8b^4x^2a^4r^2 \\
& + 10a^4x^4b^2r^2 - 6b^4r^2a^2x^2y^2 - 6a^4x^2b^2r^2y^2 - 8a^4x^2b^2r^4 - 6b^4r^2a^2x^4 \\
& + 4b^4r^4a^2x^2 - 6y^4a^4b^2x^2 - 6b^4x^4a^2y^2 + 6b^6x^2r^2a^2 + 2b^4x^2y^4a^2 \\
& + 10b^4y^4a^2r^2 + 4b^6y^2r^2a^2 - 8b^4y^2r^4a^2 - 8b^4y^2a^4r^2 + 2r^6a^2b^4 + a^8b^4 \\
& - 6a^4y^4r^2b^2 + 4a^4y^2r^4b^2 - 2a^8b^2x^2 + x^4a^4y^4 + 4x^2b^6y^2r^2 - 6y^4b^6a^2 \\
& - 4a^6y^2b^4 - 2y^4a^4x^2r^2 + 2b^2r^2a^2x^4y^2 - 2r^8a^2b^2 + r^8a^4 + a^8x^4 + a^4x^8 \\
& - 4a^4x^6r^2 + 6a^4x^4r^4 + 2b^2r^2a^2x^6 - 6b^2r^4a^2x^4 - 2a^6r^6 + 2a^6x^6 + a^8r^4 \\
& + 2y^6b^2a^2x^2 - 2r^6a^4y^2 + 6r^6b^2y^2a^2 - 2a^8x^2r^2 + y^4a^4r^4 - 6y^4b^2r^4a^2 \\
& - 2a^6y^2r^4 + 4x^2a^6y^2r^2 + 6x^2a^4y^2r^4 + 2y^4b^2x^2a^2r^2 - 10x^2b^2y^2r^4a^2 \\
& + 2y^6b^2a^2r^2 + 6r^6b^2x^2a^2 + 4b^2x^4y^4a^2 + 2a^4x^6y^2 - 6a^6x^4r^2 - 4r^6a^4x^2 \\
& + 2b^2x^6a^2y^2 + 6r^4a^6x^2 - 6a^4x^4r^2y^2 + 2b^6y^6 - 2a^8b^2r^2 - 2x^2y^4b^6 \\
& + 6a^4y^4b^4 + y^4b^8 - 6y^4b^6r^2 - 2x^4a^6y^2 - 2a^6b^6 + 2b^6r^4a^2 - 2b^8r^2a^2 \\
& + 2b^4a^6r^2 - 6b^4a^4r^4 + 2r^2a^4b^6 + 2r^4a^6b^2 + 2r^6a^4b^2
\end{aligned} \tag{32}$$

We want to find now those points on the variety  $\mathbf{V} = \mathbf{V}(g)$  where the gradient  $\nabla g$  is zero. Thus,  $\mathbf{V}$  can be thought of as union of points where the gradient is zero and those points where the gradient is not zero. Connected components of  $\mathbf{V}$  where the gradient is not zero constitute the 1-dimensional manifold. We are interested in a discrete finite set  $S$  of points where the gradient is zero. These points will give the singular points of the variety  $V$ . We will show next that the critical value of  $r$  is  $r_{crit} = \frac{b^2}{a}$ . Therefore, we compute the gradient  $\nabla g = \nabla g(x, y, a, b, r) = \langle h_1, h_2, h_3, h_4, h_5 \rangle$  where

$$\begin{aligned}
h_1 = & x(2a^4b^6 - 2a^4b^2x^2y^2 - 6b^4x^2a^4 + 5b^4y^2a^4 + b^6y^4 + 6b^4y^2a^2x^2 + b^6r^4 \\
& - 2b^6y^2r^2 - a^8x^2 - 2a^6y^2r^2 + a^4y^4r^2 - 3a^4y^2r^4 + 2a^4r^6 + a^8r^2 - 3a^6r^4 \\
& - b^4r^4x^2 + b^4r^6 - 4x^2b^2a^2y^4 - 3x^4b^2a^2y^2 - x^2b^4y^4 - a^2y^6b^2 - 3x^4a^4y^2 \\
& + 6a^4x^2r^2y^2 - b^2y^4a^2r^2 + 5b^2y^2r^4a^2 - 3b^2a^2x^4r^2 + 6b^2a^2x^2r^4 + 6a^4x^4r^2 \\
& + 6a^6x^2r^2 - 6a^4x^2r^4 + 2b^4y^2x^2r^2 - 3b^4r^4y^2 + 3y^4b^4r^2 - 3b^2r^6a^2 \\
& - 2b^2a^2x^2r^2y^2 + 3y^4a^4b^2 + a^8b^2 - 2a^6b^2r^2 + 4a^4b^2r^4 + 4a^4b^4r^2 - 3a^6x^4 \\
& - b^4y^6 + 3b^4y^2a^2r^2 - 10a^4b^2x^2r^2 + 3a^4b^2y^2r^2 + 6b^4x^2a^2r^2 + 6x^2a^6b^2 \\
& - 3a^6b^2y^2 - 3b^6y^2a^2 - y^4b^4a^2 - 3b^6r^2a^2 - 2b^4r^4a^2 + 2x^2y^2a^6 + 6x^4a^4b^2 \\
& - 3a^6b^4 - a^4y^4x^2 - 2x^6a^4)
\end{aligned} \tag{33}$$

---

<sup>6</sup>We assume  $a > b > 0$ .

$$\begin{aligned}
h_2 = & y(3a^2b^4x^4 - 3a^4b^6 + 6a^4b^2x^2y^2 + 5b^4x^2a^4 - 6b^4y^2a^4 + 2b^6y^2x^2 - 3b^6y^4 \\
& - 2b^4y^2a^2x^2 - 3b^6r^4 + 6b^6y^2r^2 - a^4y^2r^4 + a^4r^6 + a^6r^4 + x^4b^4r^2 - 3b^4r^4x^2 \\
& - 2b^6x^2r^2 + b^8r^2 + 2b^4r^6 - 3x^2b^2a^2y^4 - 4x^4b^2a^2y^2 - x^6b^2a^2 - 3x^2b^4y^4 \\
& - x^4b^4y^2 - x^4a^4y^2 + 2a^4x^2r^2y^2 - 3b^2y^4a^2r^2 + 6b^2y^2r^4a^2 - b^2a^2x^4r^2 \\
& + 5b^2a^2x^2r^4 + 3a^4x^4r^2 - 2a^6x^2r^2 - 3a^4x^2r^4 + 6b^4y^2x^2r^2 - 6b^4r^4y^2 \\
& + 6y^4b^4r^2 - 3b^2r^6a^2 - 2b^2a^2x^2r^2y^2 + b^8a^2 - 3a^6b^2r^2 - 2a^4b^2r^4 \\
& + 4a^4b^4r^2 + a^6x^4 - b^8y^2 - 2b^4y^6 - 10b^4y^2a^2r^2 + 3a^4b^2x^2r^2 + 6a^4b^2y^2r^2 \\
& + 3b^4x^2a^2r^2 - 3x^2a^6b^2 - 3x^2b^6a^2 + 6b^6y^2a^2 + 6y^4b^4a^2 - 2b^6r^2a^2 \\
& + 4b^4r^4a^2 - x^4a^4b^2 + 2a^6b^4 - x^6a^4)
\end{aligned} \tag{34}$$

$$\begin{aligned}
h_3 = & 6a^2b^4x^4 + 3a^4b^6 - 9a^4b^2x^2y^2 - 9b^4x^2a^4 + 6b^4y^2a^4 - 3b^6y^2x^2 + 3b^6y^4 \\
& + 10b^4y^2a^2x^2 - b^6r^4 - 2b^6y^2r^2 + 3a^4y^2r^4 + 3a^4r^6 - 2a^6r^4 + 3x^4b^4r^2 \\
& - 2b^4r^4x^2 - 3b^6x^2r^2 + b^8r^2 - b^4r^6 + 6x^2b^2a^2y^4 - 2x^4b^2a^2y^2 + 4x^6b^2a^2 \\
& - x^2b^4y^4 + 3x^4b^4y^2 - a^2r^8 - 3r^6x^2b^2 - 3r^6b^2y^2 + 5x^2b^2y^2r^4 - x^4b^2r^2y^2 \\
& - x^2b^2y^4r^2 - a^2x^8 - x^6b^2y^2 - 2x^4b^2y^4 - 6a^2x^2y^2r^4 - y^6b^2r^2 + 3y^4b^2r^4 \\
& + 6a^2x^4r^2y^2 + 2a^2y^4x^2r^2 - b^2x^2y^6 - x^6b^2r^2 + 3x^4b^2r^4 + 3x^4a^4y^2 \\
& - 6a^4x^2r^2y^2 + 6b^2y^4a^2r^2 - 4b^2y^2r^4a^2 - 10b^2a^2x^4r^2 + 8b^2a^2x^2r^4 \\
& + 9a^4x^4r^2 + 4a^6x^2r^2 - 9a^4x^2r^4 + 3b^4y^2x^2r^2 + 4b^4r^4y^2 - 5y^4b^4r^2 \\
& - 2b^2r^6a^2 + 6b^2a^2x^2r^2y^2 - b^8a^2 + 4a^6b^2r^2 - 3a^4b^2r^4 - 3a^4b^4r^2 - 2a^6x^4 \\
& + b^8y^2 + 2b^4y^6 + 8b^4y^2a^2r^2 - 6a^4b^2x^2r^2 - 9a^4b^2y^2r^2 + 8b^4x^2a^2r^2 \\
& + 4x^2a^6b^2 + 4x^2b^6a^2 - 6b^6y^2a^2 - 6y^4b^4a^2 - 2b^6r^2a^2 + 6b^4r^4a^2 \\
& + 9x^4a^4b^2 - 2a^6b^4 - 3x^6a^4 + 4a^2x^6r^2 + 4a^2r^6x^2 - a^2y^4x^4 - 6a^2x^4r^4 \\
& - 2a^2x^6y^2 - a^2y^4r^4 + 2a^2r^6y^2 + r^8b^2
\end{aligned} \tag{35}$$



$$\begin{aligned}
h_4 = & 2a^4b^6 - 10a^4b^2x^2y^2 - 6b^4x^2a^4 + 9b^4y^2a^4 + 2b^6y^4 + 9b^4y^2a^2x^2 + 2b^6r^4 \\
& - 4b^6y^2r^2 - a^8x^2 + 3a^6y^2r^2 - 3a^4y^4r^2 + 2a^4y^2r^4 + a^4r^6 - a^8r^2 + a^6r^4 \\
& - 3b^4r^4x^2 - 3b^4r^6 + 2x^2b^2a^2y^4 - 6x^4b^2a^2y^2 - 3x^2b^4y^4 - 4a^2y^6b^2 - a^2r^8 \\
& - 2r^6x^2b^2 - 4r^6b^2y^2 + 6x^2b^2y^2r^4 - 2x^4b^2r^2y^2 - 6x^2b^2y^4r^2 + x^4b^2y^4 \\
& - 5a^2x^2y^2r^4 - 4y^6b^2r^2 + 6y^4b^2r^4 + a^2x^4r^2y^2 + a^2y^4x^2r^2 + 2b^2x^2y^6 \\
& + x^4b^2r^4 + x^4a^4y^2 - 3a^4x^2r^2y^2 + 10b^2y^4a^2r^2 - 8b^2y^2r^4a^2 - 6b^2a^2x^4r^2 \\
& + 4b^2a^2x^2r^4 + 5a^4x^4r^2 + 2a^6x^2r^2 - 4a^4x^2r^4 + 6b^4y^2x^2r^2 + 9b^4r^4y^2 \\
& - 9y^4b^4r^2 + 2b^2r^6a^2 - 6b^2a^2x^2r^2y^2 + 6y^4a^4b^2 + a^8b^2 + 2a^6b^2r^2 \\
& - 6a^4b^2r^4 + 3a^4b^4r^2 - 3a^6x^4 + 3b^4y^6 + 6b^4y^2a^2r^2 - 8a^4b^2x^2r^2 \\
& - 8a^4b^2y^2r^2 + 9b^4x^2a^2r^2 + b^2y^8 + a^2y^6r^2 + x^2a^2y^6 + 6x^2a^6b^2 - 4a^6b^2y^2 \\
& - 4b^6y^2a^2 - 9y^4b^4a^2 - 4b^6r^2a^2 + 3b^4r^4a^2 + 3x^2y^2a^6 + 6x^4a^4b^2 - 3a^6b^4 \\
& - 3a^4y^4x^2 - 2x^6a^4 + a^2x^6r^2 + 3a^2r^6x^2 + 2a^2y^4x^4 - 3a^2x^4r^4 + a^2x^6y^2 \\
& - 3a^2y^4r^4 + 3a^2r^6y^2 + r^8b^2
\end{aligned} \tag{36}$$

$$\begin{aligned}
h_5 = & 3a^2b^4x^4 + a^4b^6 - 3a^4b^2x^2y^2 - 4b^4x^2a^4 - 4b^4y^2a^4 + 2b^6y^2x^2 - 3b^6y^4 \\
& - 3b^4y^2a^2x^2 - 3b^6r^4 + 6b^6y^2r^2 - a^8x^2 - 2a^6y^2r^2 + a^4y^4r^2 - 3a^4y^2r^4 \\
& + 2a^4r^6 + a^8r^2 - 3a^6r^4 + x^4b^4r^2 - 3b^4r^4x^2 - 2b^6x^2r^2 + b^8r^2 + 2b^4r^6 \\
& + x^2b^2a^2y^4 + x^4b^2a^2y^2 + x^6b^2a^2 - 3x^2b^4y^4 - x^4b^4y^2 + a^2y^6b^2 - 3x^4a^4y^2 \\
& + 6a^4x^2r^2y^2 - 6b^2y^4a^2r^2 + 9b^2y^2r^4a^2 - 6b^2a^2x^4r^2 + 9b^2a^2x^2r^4 \\
& + 6a^4x^4r^2 + 6a^6x^2r^2 - 6a^4x^2r^4 + 6b^4y^2x^2r^2 - 6b^4r^4y^2 + 6y^4b^4r^2 \\
& - 4b^2r^6a^2 - 10b^2a^2x^2r^2y^2 - 3y^4a^4b^2 - a^8b^2 - b^8a^2 + 2a^6b^2r^2 + 3a^4b^2r^4 \\
& - 6a^4b^4r^2 - 3a^6x^4 - b^8y^2 - 2b^4y^6 - 8b^4y^2a^2r^2 - 8a^4b^2x^2r^2 + 4a^4b^2y^2r^2 \\
& + 4b^4x^2a^2r^2 + 2x^2a^6b^2 + 3x^2b^6a^2 + 3a^6b^2y^2 + 2b^6y^2a^2 + 5y^4b^4a^2 \\
& + 2b^6r^2a^2 + 3b^4r^4a^2 + 2x^2y^2a^6 + 5x^4a^4b^2 + a^6b^4 - a^4y^4x^2 - 2x^6a^4
\end{aligned} \tag{37}$$

are polynomials in  $\mathbb{R}[a, y, a, b, r]$ . Continuing our approach from Section 2, in order to find solutions to the vector equation  $\nabla g = 0$  in  $\mathbf{V}(g)$ , we need to solve a system of six polynomial equations  $h_i = 0, i = 1, \dots, 5$ , and  $g = 0$  in the variables  $x, y, a, b, r$  under the assumption that  $a > b > r > 0$ . Let  $I_S = \langle \nabla g, g \rangle = \langle h_1, h_2, h_3, h_4, h_5, g \rangle$  and let  $F$  be a list containing the six polynomial equations:

$$F = [h_1 = 0, h_2 = 0, h_3 = 0, h_4 = 0, h_5 = 0, g = 0] \tag{38}$$

In order to solve system (38), we can either employ the Gröbner basis approach again (see Appendix C) or simply use `solve` command in **Maple**. Thus we find that the affine variety  $\mathbf{V}(I_S)$  consists of the following eight singular points of which six are real and two are complex:

$$S_1 = \left( 0, \frac{\sqrt{-(a^2 - b^2)(r^2 - b^2)}}{b} \right), \quad S_2 = \left( 0, -\frac{\sqrt{-(a^2 - b^2)(r^2 - b^2)}}{b} \right), \tag{39}$$

$$S_3 = \left( \frac{\sqrt{(a^2 - b^2)(-a^2 + r^2)}}{a}, 0 \right), \quad S_4 = \left( -\frac{\sqrt{(a^2 - b^2)(-a^2 + r^2)}}{a}, 0 \right), \quad (40)$$

$$S_5 = \left( \sqrt{-\frac{\delta}{(-a^2 + b^2)(abr)^{\frac{2}{3}}}}, -\frac{\sqrt{(-a^2 + b^2)\varepsilon}}{(abr)^{\frac{1}{3}}(-a^2 + b^2)} \right), \quad (41)$$

$$S_6 = \left( -\sqrt{-\frac{\delta}{(-a^2 + b^2)(abr)^{\frac{2}{3}}}}, -\frac{\sqrt{(-a^2 + b^2)\varepsilon}}{(abr)^{\frac{1}{3}}(-a^2 + b^2)} \right), \quad (42)$$

$$S_7 = \left( \sqrt{-\frac{\delta}{(-a^2 + b^2)(abr)^{\frac{2}{3}}}}, \frac{\sqrt{(-a^2 + b^2)\varepsilon}}{(abr)^{\frac{1}{3}}(-a^2 + b^2)} \right), \quad (43)$$

$$S_8 = \left( -\sqrt{-\frac{\delta}{(-a^2 + b^2)(abr)^{\frac{2}{3}}}}, \frac{\sqrt{(-a^2 + b^2)\varepsilon}}{(abr)^{\frac{1}{3}}(-a^2 + b^2)} \right), \quad (44)$$

where

$$\delta = -3b^2a^2r^2 + r^2a^2(abr)^{\frac{2}{3}} + 3(abr)^{\frac{1}{3}}b^3ar - b^4(abr)^{\frac{2}{3}}, \quad (45)$$

$$\varepsilon = -a^4(abr)^{\frac{2}{3}} + 3(abr)^{\frac{1}{3}}bra^3 - 3b^2a^2r^2 + (abr)^{\frac{2}{3}}b^2r^2. \quad (46)$$

Notice that  $\delta$  becomes zero in three cases:

1. When  $r = 0$  – this is not possible since we expect  $r > 0$ ,
2. When  $r = -\frac{b^2}{a}$  – this is not possible as  $r > 0$ ,
3. When  $r = \frac{b^2}{a}$ .

Only this last value is possible thus we set  $r_{crit} = \frac{b^2}{a}$ . When  $r = r_{crit}$ , then under the assumptions that  $a > b > r > 0$ , we essentially obtain two real singular points:

$$S_1 = S_5 = S_6 = \left( 0, \frac{a^2 - b^2}{a} \right) \quad \text{and} \quad S_2 = S_7 = S_8 = \left( 0, -\frac{a^2 - b^2}{a} \right), \quad (47)$$

while the complex points  $S_3$  and  $S_4$  become:

$$S_3 = \left( \frac{\sqrt{a^2 + b^2}(a^2 - b^2)}{a^2}i, 0 \right) \quad \text{and} \quad S_4 = \left( -\frac{\sqrt{a^2 + b^2}(a^2 - b^2)}{a^2}i, 0 \right). \quad (48)$$

For the ellipse with  $a > b$ , the ratio  $l = \frac{b^2}{a}$  is again called a *semi-latus rectum*. It is the distance between a focus and the ellipse itself measured along a line perpendicular to the major axis. The value of  $l$  is also the reciprocal of the maximum curvature  $\kappa_{max}$  of the ellipse, or, it is equal to the radius of the smallest osculating circle. [14]

To summarize, the variety  $\mathbf{V}(I_S)$  contains the following real singular points:

1. Two virtual points  $S_1$  and  $S_2$  shown in (39) when  $0 < r < r_{crit}$ . In particular, when  $a = 3, b = \frac{3}{2}$ , and  $r = \frac{1}{2}$ , upon substituting these values into (32) we obtain the parallel lines (26) derived in Example 4. Furthermore, when these values are substituted into (39), then we obtain the two virtual singular points found in the same example.

2. Two points  $S_1$  and  $S_2$  shown in (47) when  $r = r_{crit}$ . In particular, we can recover results of Example 5 when we substitute values  $a = 3$ ,  $b = \frac{3}{2}$ , and  $r = \frac{3}{4}$  into the general equation (32) and the singular points (47).
3. Six points  $S_1, S_2, S_5, S_6, S_7, S_8$  shown in (39), (41), (42), (43), and (44) when  $r > r_{crit}$ . In particular we can recover in the same manner six points and the equation for the parallel lines found in Example 6 when we set  $a = 3$ ,  $b = \frac{3}{2}$ , and  $r = \frac{3}{2}$ . These six singular points are displayed in Figure 6.

## 4 Parallel Lines to a Hyperbola

Since a parabola, an ellipse, and a hyperbola belong to the same class of nondegenerate conic

$$C : (X^2 + Y^2 - Z^2 = 0)$$

in  $\mathbb{P}^2(\mathbb{R})$ , (cf. [12]) we expect essentially similar results for a hyperbola. Thus, our approach will be similar to the one for the ellipse from Section 3. In particular, there is only one sign change needed in the polynomials  $f_1$  and  $f_2$  shown in (23) and (25) whereas the polynomial  $f_2$  remains the same. Thus, we have now

$$f_1 = b^2 y_0^2 - a^2 x_0^2 - a^2 b^2 = 0 \quad (49)$$

where we assume that  $a > b$  and the center of the hyperbola is at the origin. Polynomial  $f_2$  defines a circle of radius (offset)  $r$  centered at a point  $P = (x_0, y_0)$  on the hyperbola,

$$f_2 = (y - y_0)^2 + (x - x_0)^2 - r^2 = 0 \quad (50)$$

while polynomial  $f_3$

$$f_3 = b^2(x - x_0)y_0 + a^2(y - y_0)x_0 = 0 \quad (51)$$

gives a condition that a point  $P(x, y)$  on the circle  $f_2$  lies on a line perpendicular to the hyperbola  $f_1$  and passing through the point  $(x_0, y_0)$  on the hyperbola. All these points  $P$  belong to the affine variety  $\mathbf{V} = \mathbf{V}(f_1, f_2, f_3)$  – the hyperbola envelope – and, like for the parabola in Section 2 and the ellipse in Section 3, define two parallel lines at the distance  $r$  from the hyperbola. We proceed like in Section 3: We will compute a reduced Gröbner basis for  $\mathbf{V}$  and by eliminating variables  $x_0, y_0$  we will obtain one polynomial  $g$  in variables  $x$  and  $y$  alone (also  $a, b, r$  in the general case) that defines both parallel lines on each side of the hyperbola for the given value of the offset  $r$ . Thus, in general, we have  $f_1, f_2, f_3 \in \mathbb{R}[y_0, x_0, x, y, a, b, r]$ . However, first we illustrate computations for specific values of the parameters  $a, b$  and  $r$ , and later we show computation for unspecified parameters.

### 4.1 Parallel lines for specific parameters $a, b, r$

The parallel lines constitute a subvariety of the affine variety  $\mathbf{V} = \mathbf{V}(I)$  where, like before,  $I = \langle f_1, f_2, f_3 \rangle$ . In order to find the polynomial  $g$  for the second elimination ideal  $I_2 = I \cap \mathbb{R}[x, y]$ , we first find a Gröbner basis  $G$  for  $\mathbf{V}$  for the lexicographic order  $y_0 > x_0 > x > y$  and reduce it eventually to a unique reduced basis  $G_r$  of nine (or eleven, depending on the case) polynomials. Finally, we eliminate from that basis all polynomials that contain variables  $y_0$  and  $x_0$ , and keep the single polynomial  $g \in \mathbb{R}[x, y]$  that gives the parallel lines and generates the second elimination ideal  $I_2 = \langle g \rangle$ . We will also proceed to find all singular points of  $\mathbf{V}(g)$  where  $(\nabla g)(p) = 0$ . Thus, let's consider three cases when  $0 < r < r_{crit}$ ,  $r = r_{crit}$ , and  $r > r_{crit}$ . In the following section we will consider a general case for arbitrary values of  $a, b$  ( $a > b > 0$ ) and  $r$  ( $r > 0$ ).

**Example 7** ( $r < r_{crit}$ ) Let  $a = 3$ ,  $b = \frac{3}{2}$ , and  $r = \frac{1}{2}$ . Then,

$$\mathbf{V} = \mathbf{V}(y_0^2 - x_0^2 - 1, 4y^2 - 8yy_0 + 4y_0^2 + 4x^2 - 8xx_0 + 4x_0^2 - 1, xy_0 + yx_0 - 2y_0x_0)$$

We find that the reduced Gröbner basis  $G_r$  contains nine polynomials including

$$g = 225 + 64y^8 + 288x^6 + 480x^2y^4 - 800y^2x^4 - 480y^6 - 128y^4x^4 - 1180y^2 + 516x^2 - 1320x^2y^2 + 532x^4 + 1236y^4 + 64x^8 \quad (52)$$

whose gradient is

$$\begin{aligned} \nabla g = & x(216x^4 + 120y^4 - 400x^2y^2 - 64x^2y^4 + 129 - 330y^2 + 266x^2 + 64x^6)\mathbf{i} \\ & - y(-64y^6 - 240x^2y^2 + 200x^4 + 360y^4 + 64y^2x^4 + 295 + 330x^2 - 618y^2)\mathbf{j}. \end{aligned} \quad (53)$$

There are exactly two solutions to the equation  $(\nabla g)(p) = 0$  on  $\mathbf{V}(g)$  :

$$S_1 = \left(0, \frac{\sqrt{10}}{2}\right) \quad \text{and} \quad S_2 = \left(0, -\frac{\sqrt{10}}{2}\right).$$

We will refer to these points as virtual singular points for the following reason:  $\mathbf{V}(g)$  is a disjoint union of  $\{S_1, S_2\}$  and a non-singular smooth subvariety  $\Pi$  where the gradient  $\nabla g$  does not vanish. The subvariety  $\Pi$  constitutes the set of two parallel lines that we have been seeking. Note that the foci  $F_1 = (0, \sqrt{2})$  and  $F_2 = (0, -\sqrt{2})$  are different from the singular points. We show all in Figure 7.

We have verified by direct computation that as the value of the offset  $r$  increases towards some critical value  $r_{crit}$ , the virtual singular points approach the smooth component of the variety  $\mathbf{V}(g)$ , that is, they approach the parallel lines  $\Pi$ .<sup>7</sup> As the next example shows, when  $r = r_{crit}$ , then the virtual singular points are actually located on the smooth component  $\Pi$  of the variety  $\mathbf{V}(g)$ .

**Example 8** ( $r = r_{crit}$ ) Let  $a = \frac{3}{2}$ ,  $b = 1$ , and  $r = \frac{2}{3}$ . Then,

$$\mathbf{V} = \mathbf{V}(4y_0^2 - 9x_0^2 - 9, 9y^2 - 18yy_0 + 9y_0^2 + 9x^2 - 18xx_0 + 9x_0^2 - 4, 4xy_0 - 13x_0y_0 + 9x_0y)$$

We find that the reduced Gröbner basis  $G_r$  contains eleven polynomials including

$$\begin{aligned} g = & 9447840x^6y^2 - 4199040y^6x^2 + 53800200x^6 + 1679616y^8 + 124857369x^4 \\ & + 127471968y^4 - 159339960x^2y^2 - 24820992y^6 + 46539360y^4x^2 - 105471720y^2x^4 \\ & - 4933872y^4x^4 + 120670225 + 8503056x^8 - 250879824y^2 + 156799890x^2 \end{aligned} \quad (54)$$

whose gradient is

$$\begin{aligned} \nabla g = & x(1574640y^2x^4 - 233280y^6 + 8966700x^4 + 13873041x^2 - 8852220y^2 + 2585520y^4 \\ & - 11719080x^2y^2 - 548208y^4x^2 + 1889568x^6 + 8711105)\mathbf{i} \\ & + y(131220x^6 - 174960y^4x^2 + 93312y^6 + 3540888y^2 - 2213055x^2 - 1034208y^4 \\ & + 1292760x^2y^2 - 1464885x^4 - 137052y^2x^4 - 3484442)\mathbf{j}. \end{aligned} \quad (55)$$

---

<sup>7</sup>In the next subsection we will derive a formula for the critical value  $r_{crit}$  of  $r$  for a general hyperbola.

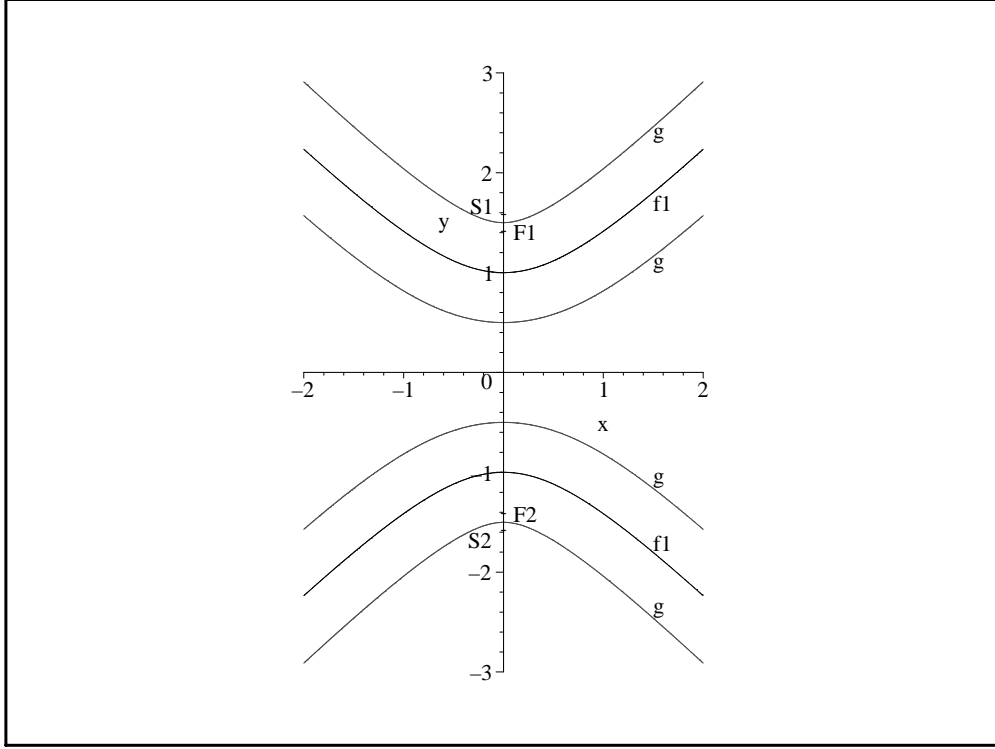


Figure 7: Hyperbola with parallel lines  $g$ , foci  $F_1, F_2$  and two singular points  $S_1, S_2$  for  $a = 1, b = 1$  and  $r = \frac{1}{2}$

There are exactly two solutions to the equation  $(\nabla g)(p) = 0$  on  $\mathbf{V}(g)$  :

$$S_1 = \left(0, \frac{13}{6}\right) \quad \text{and} \quad S_2 = \left(0, -\frac{13}{6}\right)$$

These points are the singular points of the variety  $\mathbf{V}(g)$  : At every other point of this variety the gradient  $\nabla g$  does not vanish. We will collect these points as before into a non-singular subvariety  $\Pi$ . The variety  $\Pi$  constitutes the set of two parallel lines that we have been seeking. Note that the foci  $F_1 = (0, \frac{\sqrt{13}}{2})$  and  $F_2 = (0, -\frac{\sqrt{13}}{2})$  are still different from the singular points. We show all in Figure 8.

As we can see from the last figure, when  $r = r_{crit} = \frac{2}{3}$  for the given values of  $a$  and  $b$ , the singular points are located on the continuous component of  $\Pi$ .

Probably the most interesting parallel lines we again obtain when  $r > r_{crit}$  as we show in the next example. There, we will find six singular points as each of the two singular points found in Example 8 splits into three points.

**Example 9** ( $r > r_{crit}$ ) Let  $a = \frac{3}{2}$ ,  $b = 1$ , and  $r = \frac{4}{3}$ . Then,

$$\mathbf{V} = \mathbf{V}(4y_0^2 - 9x_0^2 - 9, 9y^2 - 18yy_0 + 9y_0^2 + 9x^2 - 18xx_0 + 9x_0^2 - 16, 4xy_0 - 13x_0y_0 + 9x_0y)$$

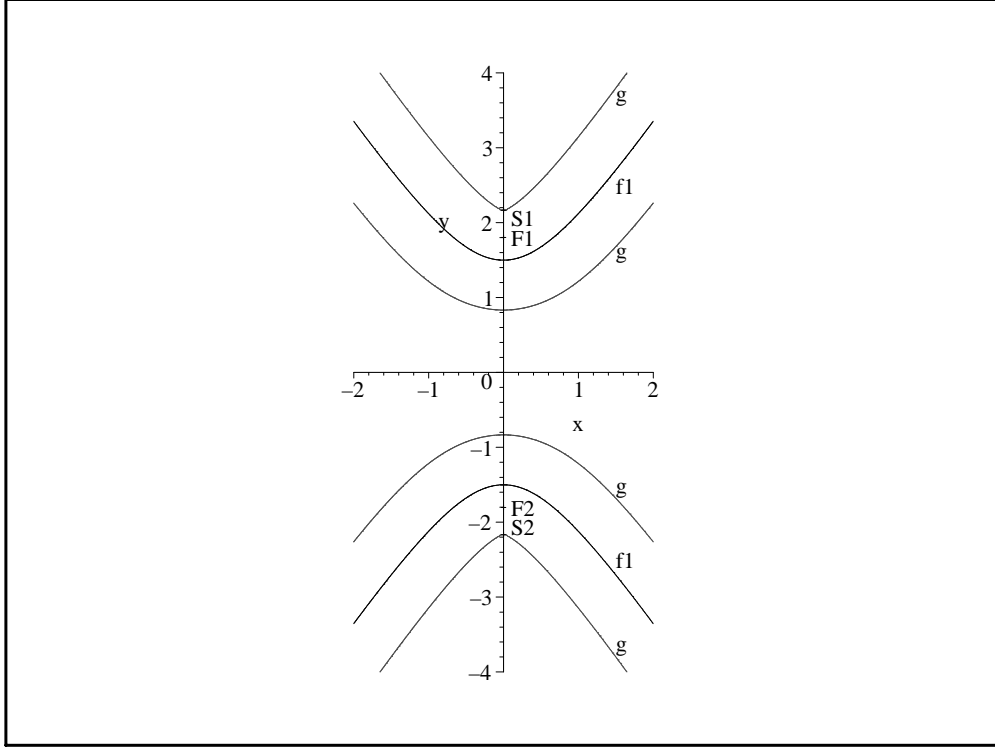


Figure 8: Hyperbola with parallel lines  $g$ , foci  $F_1, F_2$  and singular points  $S_1, S_2$  for  $a = \frac{3}{2}$ ,  $b = 1$  and  $r = \frac{2}{3}$

We find that the reduced Gröbner basis  $G_r$  again contains nine polynomials including

$$\begin{aligned}
 g = & -1627128x^6 + 8503056x^8 + 30525625 + 83351034x^2 - 1109487600y^2 + 264529800x^2y^2 \\
 & + 51521913x^4 + 381560544y^4 + 349920y^4x^2 - 188052840y^2x^4 - 43856640y^6 - 4933872y^4x^4 \\
 & + 9447840x^6y^2 - 4199040y^6x^2 + 1679616y^8 \quad (56)
 \end{aligned}$$

whose gradient is

$$\begin{aligned}
 \nabla g = & x(-271188x^4 + 1889568x^6 + 4630613 + 14696100y^2 + 5724657x^2 + 19440y^4 \\
 & - 20894760x^2y^2 - 548208y^4x^2 + 1574640y^2x^4 - 233280y^6)\mathbf{i} \\
 & + y(-15409550 + 3674025x^2 + 10598904y^2 + 9720x^2y^2 - 2611845x^4 - 1827360y^4 \\
 & - 137052y^2x^4 + 131220x^6 - 174960y^4x^2 + 93312y^6)\mathbf{j}. \quad (57)
 \end{aligned}$$

There are exactly six solutions to the equation  $(\nabla g)(p) = 0$  on  $\mathbf{V}(g)$ , or, in another words, six singular points of the variety:

$$S_1 = \left(0, \frac{5\sqrt{13}}{6}\right) \quad \text{and} \quad S_2 = \left(0, -\frac{5\sqrt{13}}{6}\right)$$

$$\begin{aligned}
S_3 &= \left( \frac{2 \sqrt{39 - 78 \cdot 2^{\frac{1}{3}} + 39 \cdot 2^{\frac{2}{3}}}}{13}, \frac{\sqrt{12805 + 11232 \cdot 2^{\frac{1}{3}} + 12636 \cdot 2^{\frac{2}{3}}}}{78} \right), \\
S_4 &= \left( -\frac{2 \sqrt{39 - 78 \cdot 2^{\frac{1}{3}} + 39 \cdot 2^{\frac{2}{3}}}}{13}, \frac{\sqrt{12805 + 11232 \cdot 2^{\frac{1}{3}} + 12636 \cdot 2^{\frac{2}{3}}}}{78} \right), \\
S_5 &= \left( \frac{2 \sqrt{39 - 78 \cdot 2^{\frac{1}{3}} + 39 \cdot 2^{\frac{2}{3}}}}{13}, -\frac{\sqrt{12805 + 11232 \cdot 2^{\frac{1}{3}} + 12636 \cdot 2^{\frac{2}{3}}}}{78} \right), \\
S_6 &= \left( -\frac{2 \sqrt{39 - 78 \cdot 2^{\frac{1}{3}} + 39 \cdot 2^{\frac{2}{3}}}}{13}, -\frac{\sqrt{12805 + 11232 \cdot 2^{\frac{1}{3}} + 12636 \cdot 2^{\frac{2}{3}}}}{78} \right).
\end{aligned}$$

We display these six points in Figure 9.

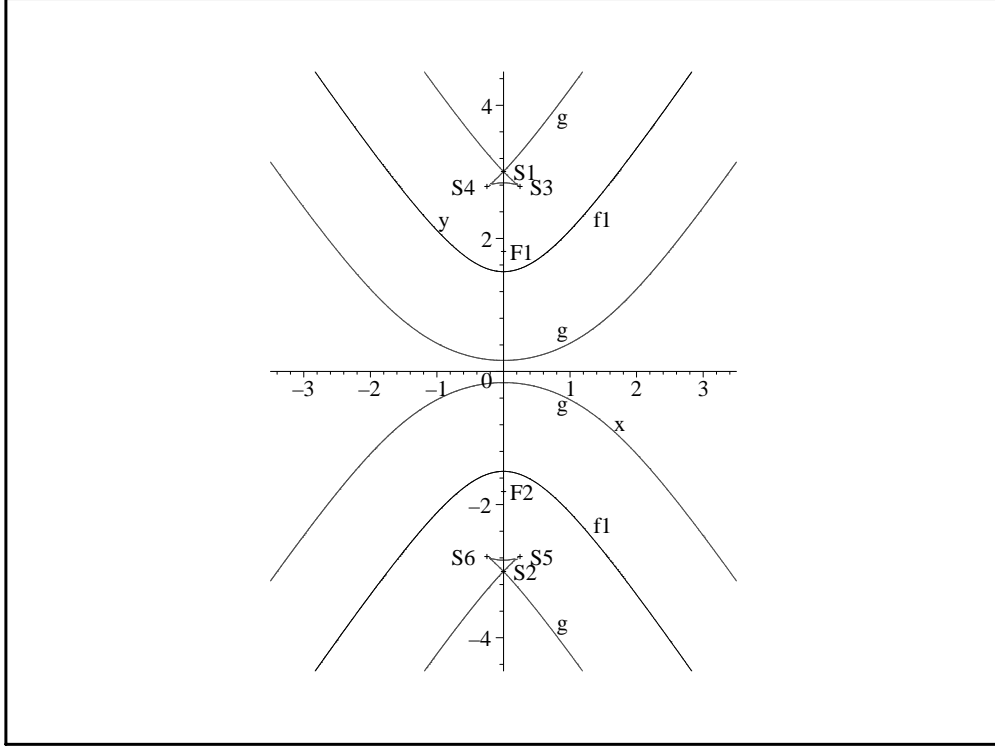


Figure 9: Hyperbola with parallel lines  $g$ , foci  $F_1, F_2$  and six singular points  $S_i, i = 1, \dots, 6$ , for  $a = \frac{3}{2}, b = 1$  and  $r = \frac{4}{3}$

#### 4.2 Parallel lines for arbitrary parameters $a, b, r$

In this section we will find a general polynomial  $g$  that will generate the second elimination ideal  $I_2 = I \cap \mathbb{R}[x, y, a, b, r]$  where  $I = \langle f_1, f_2, f_3 \rangle$  and the generating polynomials are defined in equations (49), (50),

and (51). We will also find a formula for the critical value  $r_{crit}$  of  $r$  that determines whether  $\mathbf{V}(I_2)$  has two or six singular points, and, we will find general formulas for the coordinates of the said singular points for arbitrary values of the parameters  $a, b$  and  $r$ .

We begin with a computation of a Gröbner basis for the ideal  $I$  for the lexicographical order  $y_0 > x_0 > x > y > a > b > r$ .

$$I = \langle b^2 y_0^2 - a^2 x_0^2 - a^2 b^2, y^2 - 2yy_0 + y_0^2 + x^2 - 2xx_0 + x_0^2 - r^2, b^2 y_0 x - b^2 y_0 x_0 + a^2 x_0 y - a^2 x_0 y_0 \rangle$$

We found that a reduced Gröbner basis for  $I$  consists of fifteen homogeneous polynomials of degrees: 16 (two polynomials), 14 (one polynomial), 12 (one polynomial), 11 (two polynomials), 10 (two polynomials), 9 (one polynomial), 8 (one polynomial), 7 (two polynomials), 5 (one polynomial), 4 (one polynomial), and 2 (one polynomial). We will not display these polynomials as they are similar to the polynomials for the ellipse shown in Appendix B.

Thus, in this general case we observe that  $I_2 = \langle g \rangle$  where  $g = g_1/(a^2 b^2)$  is the following homogeneous polynomial in  $\mathbb{R}[x, y, a, b, r]$  of degree twelve:

$$\begin{aligned} g = & 6a^4 x^4 r^4 - 2b^2 x^4 a^2 r^2 y^2 + 4b^6 x^2 a^4 + 2a^8 b^2 x^2 + 2a^6 x^6 + 6y^4 a^2 b^6 - 2a^4 b^2 x^4 y^2 \\ & + 4a^4 x^6 b^2 - 6a^6 x^2 b^2 y^2 - 4a^6 x^2 b^2 r^2 - 10a^4 x^2 b^4 y^2 - 8a^4 x^2 b^4 r^2 \\ & + 8a^4 x^2 b^2 r^4 - 6b^4 r^2 a^2 x^4 + 4b^4 r^4 a^2 x^2 - 10a^4 x^4 b^2 r^2 - 6b^4 r^2 a^2 x^2 y^2 \\ & + 6a^4 x^2 b^2 r^2 y^2 - 6b^4 x^4 a^2 y^2 + 2b^4 x^2 y^4 a^2 - 6b^6 x^2 r^2 a^2 + 10b^4 y^4 a^2 r^2 \\ & - 8b^4 y^2 a^4 r^2 - 4b^6 y^2 r^2 a^2 - 8b^4 y^2 r^4 a^2 + 6y^4 a^4 b^2 x^2 - 2a^2 y^2 b^8 + 6b^4 y^4 a^4 \\ & - 4b^4 y^6 a^2 - 6y^2 a^4 b^6 + 6a^4 x^4 b^4 - 4y^2 a^6 b^4 - 6y^2 a^6 r^2 b^2 + 6y^4 a^4 r^2 b^2 \\ & - 4y^2 a^4 r^4 b^2 + 2a^6 b^6 + a^4 b^8 + 6a^6 b^4 x^2 + 6a^6 x^4 b^2 + b^8 r^4 + a^8 b^4 - 2a^2 b^8 r^2 \\ & - 2a^2 b^6 r^4 + 2b^6 x^2 r^4 - 4b^6 x^2 y^2 r^2 + 6b^2 x^4 a^2 r^4 + 4a^6 x^2 r^2 y^2 - 2b^2 x^2 a^2 y^6 \\ & + 2b^4 x^2 y^6 - 2b^2 x^2 y^4 a^2 r^2 + 6b^4 x^2 y^2 r^4 + 10b^2 x^2 y^2 a^2 r^4 + 2b^6 x^2 y^4 \\ & - 6b^4 x^2 y^4 r^2 + a^4 x^8 + 2b^6 r^6 + b^4 r^8 + 6a^6 x^2 r^4 - 6a^6 x^4 r^2 - 2a^8 x^2 r^2 \\ & - 6a^4 b^4 r^4 - 2a^4 b^6 r^2 - 2a^6 y^2 r^4 - 2a^6 b^2 r^4 - 6a^2 r^6 b^2 y^2 - 2a^4 r^6 y^2 \\ & - 6a^4 x^4 r^2 y^2 + 6a^2 y^4 b^2 r^4 + 2a^8 b^2 r^2 + a^4 y^4 r^4 - 2a^2 y^6 b^2 r^2 - 4a^4 r^6 x^2 \\ & + a^8 x^4 + 6a^4 x^2 y^2 r^4 - 4a^4 x^6 r^2 + a^8 r^4 + 2a^4 x^6 y^2 - 2a^6 x^4 y^2 - 2a^6 r^6 \\ & - 2a^4 r^6 b^2 + 2a^6 r^2 b^4 + a^4 y^4 x^4 + 2a^2 r^8 b^2 - 2a^4 y^4 x^2 r^2 + 2a^2 b^4 r^6 + a^4 r^8 \\ & + 6y^4 b^4 r^4 + 6y^4 b^6 r^2 - 4y^6 b^4 r^2 - 4y^2 b^4 r^6 - 6y^2 b^6 r^4 + y^4 b^8 + y^8 b^4 - 2y^6 b^6 \\ & - 2b^2 x^6 a^2 y^2 - 2b^4 x^2 r^6 - 6b^2 x^2 r^6 a^2 + b^4 x^4 y^4 - 4b^2 x^4 y^4 a^2 + b^4 x^4 r^4 \\ & - 2b^2 x^6 a^2 r^2 - 2b^4 x^4 r^2 y^2 - 2y^2 b^8 r^2 - 6y^2 x^2 a^2 b^6 \end{aligned} \tag{58}$$

We want to find now those points on the variety  $\mathbf{V} = \mathbf{V}(g)$  where the gradient  $\nabla g$  is zero. Thus,  $\mathbf{V}$  can be thought of as union of points where the gradient is zero and those points where the gradient is not zero. Connected components of  $\mathbf{V}$  where the gradient is not zero constitute the 1-dimensional manifold. We are interested in a discrete finite set  $S$  of points where the gradient is zero. These points will give the singular points of the variety  $V$ . We will show next that the critical value of  $r$  is  $r_{crit} = \frac{b^2}{a}$ . Therefore, we compute



$\nabla g = \nabla g(x, y, a, b, r) = \langle h_1, h_2, h_3, h_4, h_5 \rangle$  where

$$\begin{aligned}
h_1 = & x(2b^6y^2r^2 - a^8b^2 + b^4r^6 - x^2b^4r^4 + 2x^2b^4r^2y^2 - 3b^4y^2r^4 + 3b^4y^4r^2 - b^6y^4 \\
& - 3a^6b^4 - a^8x^2 - 2a^4b^6 - 3a^4y^4b^2 + 3y^2a^6b^2 + 3y^2b^6a^2 + 5b^4y^2a^4 \\
& - b^4y^4a^2 - 6a^4x^2b^4 + r^2a^8 - b^4x^2y^4 + 2a^6x^2y^2 + 6b^4x^2a^2y^2 + 2a^4b^2x^2y^2 \\
& - b^4y^6 - b^6r^4 + 10a^4x^2b^2r^2 - 6a^6x^2b^2 - 6a^4x^4b^2 - 3a^6x^4 + 3b^6r^2a^2 \\
& - 2b^4r^4a^2 + 2r^6a^4 + 3r^6b^2a^2 - 3r^4a^6 + 4r^2a^4b^4 + 2b^2r^2a^2x^2y^2 \\
& + 6a^4x^2r^2y^2 + 6a^6x^2r^2 + 3b^2r^2a^2x^4 - 6a^4x^2r^4 + b^2y^4a^2r^2 - 5b^2y^2r^4a^2 \\
& - 3y^2a^4r^4 - 6b^2r^4a^2x^2 + 3b^2x^4a^2y^2 + 4b^2x^2y^4a^2 - y^4a^4x^2 - 3a^4x^4y^2 \\
& + 6a^4x^4r^2 - 2y^2a^6r^2 + y^4a^4r^2 + b^2y^6a^2 + 3b^4y^2a^2r^2 - 3b^2y^2a^4r^2 \\
& + 2b^2a^6r^2 + 6b^4x^2a^2r^2 - 4b^2a^4r^4 - 2a^4x^6)
\end{aligned} \tag{59}$$

$$\begin{aligned}
h_2 = & y(-6b^6y^2r^2 + 2b^4r^6 - 3x^2b^4r^4 + 6x^2b^4r^2y^2 - 6b^4y^2r^4 + 6b^4y^4r^2 + r^2b^8 \\
& + b^4x^4r^2 + 2x^2b^6r^2 + 3b^6y^4 + 2a^6b^4 + 3a^4b^6 - 6y^2b^6a^2 - 6b^4y^2a^4 \\
& + 6b^4y^4a^2 + b^2x^6a^2 + 3b^6x^2a^2 + 3b^4a^2x^4 + 5a^4x^2b^4 + a^2b^8 - 2y^2x^2b^6 \\
& - 3b^4x^2y^4 - b^4x^4y^2 - 2b^4x^2a^2y^2 - 6a^4b^2x^2y^2 - 2b^4y^6 + 3b^6r^4 \\
& - 3a^4x^2b^2r^2 - y^2b^8 + 3a^6x^2b^2 + a^4x^4b^2 + a^6x^4 + 2b^6r^2a^2 + 4b^4r^4a^2 + r^6a^4 \\
& + 3r^6b^2a^2 + r^4a^6 + 4r^2a^4b^4 + 2b^2r^2a^2x^2y^2 + 2a^4x^2r^2y^2 - 2a^6x^2r^2 \\
& + b^2r^2a^2x^4 - 3a^4x^2r^4 + 3b^2y^4a^2r^2 - 6b^2y^2r^4a^2 - y^2a^4r^4 - 5b^2r^4a^2x^2 \\
& + 4b^2x^4a^2y^2 + 3b^2x^2y^4a^2 - a^4x^4y^2 + 3a^4x^4r^2 - 10b^4y^2a^2r^2 - 6b^2y^2a^4r^2 \\
& + 3b^2a^6r^2 + 3b^4x^2a^2r^2 + 2b^2a^4r^4 - a^4x^6)
\end{aligned} \tag{60}$$

$$\begin{aligned}
h_3 = & 2b^2x^4y^4 + b^2x^6y^2 + b^2x^2y^6 + 2b^6y^2r^2 - b^4r^6 - 2x^2b^4r^4 + 3x^2b^4r^2y^2 \\
& + 4b^4y^2r^4 - 5b^4y^4r^2 + r^2b^8 + 3b^4x^4r^2 + 3x^2b^6r^2 - 3b^6y^4 - 2a^6b^4 - a^2x^8 \\
& + b^2x^4r^2y^2 + b^2x^2y^4r^2 - 5b^2x^2y^2r^4 - a^2y^4x^4 - r^8b^2 + 2a^2r^6y^2 - a^2y^4r^4 \\
& + 4a^2r^6x^2 - 6a^2x^4r^4 - 3b^2x^4r^4 + 6a^2x^4r^2y^2 - 3y^4b^2r^4 + 3r^6b^2y^2 + y^6b^2r^2 \\
& - 6a^2x^2y^2r^4 + 2a^2y^4x^2r^2 + 3b^2x^2r^6 + b^2x^6r^2 + 4a^2x^6r^2 - 2a^2x^6y^2 - r^8a^2 \\
& - 3a^4b^6 + 6y^2b^6a^2 + 6b^4y^2a^4 - 6b^4y^4a^2 - 4b^2x^6a^2 - 4b^6x^2a^2 - 6b^4a^2x^4 \\
& - 9a^4x^2b^4 - a^2b^8 + 3y^2x^2b^6 - b^4x^2y^4 + 3b^4x^4y^2 + 10b^4x^2a^2y^2 \\
& + 9a^4b^2x^2y^2 + 2b^4y^6 + b^6r^4 + 6a^4x^2b^2r^2 + y^2b^8 - 4a^6x^2b^2 - 9a^4x^4b^2 \\
& - 2a^6x^4 + 2b^6r^2a^2 + 6b^4r^4a^2 + 3r^6a^4 + 2r^6b^2a^2 - 2r^4a^6 - 3r^2a^4b^4 \\
& - 6b^2r^2a^2x^2y^2 - 6a^4x^2r^2y^2 + 4a^6x^2r^2 + 10b^2r^2a^2x^4 - 9a^4x^2r^4 \\
& - 6b^2y^4a^2r^2 + 4b^2y^2r^4a^2 + 3y^2a^4r^4 - 8b^2r^4a^2x^2 + 2b^2x^4a^2y^2 \\
& - 6b^2x^2y^4a^2 + 3a^4x^4y^2 + 9a^4x^4r^2 + 8b^4y^2a^2r^2 + 9b^2y^2a^4r^2 - 4b^2a^6r^2 \\
& + 8b^4x^2a^2r^2 + 3b^2a^4r^4 - 3a^4x^6
\end{aligned} \tag{61}$$

$$\begin{aligned}
h_4 = & b^2 x^4 y^4 + 2b^2 x^2 y^6 - 4b^6 y^2 r^2 + a^8 b^2 + 3b^4 r^6 + 3x^2 b^4 r^4 - 6x^2 b^4 r^2 y^2 \\
& - 9b^4 y^2 r^4 + 9b^4 y^4 r^2 + 2b^6 y^4 + 3a^6 b^4 + b^2 y^8 + a^8 x^2 - 2b^2 x^4 r^2 y^2 \\
& - 6b^2 x^2 y^4 r^2 + 6b^2 x^2 y^2 r^4 - 2a^2 y^4 x^4 + r^8 b^2 - 3a^2 r^6 y^2 + 3a^2 y^4 r^4 - 3a^2 r^6 x^2 \\
& + 3a^2 x^4 r^4 + b^2 x^4 r^4 - a^2 x^4 r^2 y^2 + 6y^4 b^2 r^4 - 4r^6 b^2 y^2 - 4y^6 b^2 r^2 + 5a^2 x^2 y^2 r^4 \\
& - a^2 y^4 x^2 r^2 - 2b^2 x^2 r^6 - a^2 x^6 r^2 - a^2 x^6 y^2 + r^8 a^2 - x^2 a^2 y^6 - a^2 y^6 r^2 + 2a^4 b^6 \\
& + 6a^4 y^4 b^2 - 4y^2 a^6 b^2 - 4y^2 b^6 a^2 - 9b^4 y^2 a^4 + 9b^4 y^4 a^2 + 6a^4 x^2 b^4 + r^2 a^8 \\
& + 3b^4 x^2 y^4 - 3a^6 x^2 y^2 - 9b^4 x^2 a^2 y^2 - 10a^4 b^2 x^2 y^2 - 3b^4 y^6 + 2b^6 r^4 \\
& - 8a^4 x^2 b^2 r^2 + 6a^6 x^2 b^2 + 6a^4 x^4 b^2 + 3a^6 x^4 - 4b^6 r^2 a^2 - 3b^4 r^4 a^2 - r^6 a^4 \\
& + 2r^6 b^2 a^2 - r^4 a^6 - 3r^2 a^4 b^4 - 6b^2 r^2 a^2 x^2 y^2 + 3a^4 x^2 r^2 y^2 - 2a^6 x^2 r^2 \\
& - 6b^2 r^2 a^2 x^4 + 4a^4 x^2 r^4 + 10b^2 y^4 a^2 r^2 - 8b^2 y^2 r^4 a^2 - 2y^2 a^4 r^4 + 4b^2 r^4 a^2 x^2 \\
& - 6b^2 x^4 a^2 y^2 + 2b^2 x^2 y^4 a^2 + 3y^4 a^4 x^2 - a^4 x^4 y^2 - 5a^4 x^4 r^2 - 3y^2 a^6 r^2 \\
& + 3y^4 a^4 r^2 - 4b^2 y^6 a^2 - 6b^4 y^2 a^2 r^2 - 8b^2 y^2 a^4 r^2 + 2b^2 a^6 r^2 - 9b^4 x^2 a^2 r^2 \\
& - 6b^2 a^4 r^4 + 2a^4 x^6
\end{aligned} \tag{62}$$

$$\begin{aligned}
h_5 = & -6b^6 y^2 r^2 + a^8 b^2 + 2b^4 r^6 - 3x^2 b^4 r^4 + 6x^2 b^4 r^2 y^2 - 6b^4 y^2 r^4 + 6b^4 y^4 r^2 + r^2 b^8 \\
& + b^4 x^4 r^2 + 2x^2 b^6 r^2 + 3b^6 y^4 + a^6 b^4 - a^8 x^2 - a^4 b^6 + 3a^4 y^4 b^2 - 3y^2 a^6 b^2 \\
& - 2y^2 b^6 a^2 - 4b^4 y^2 a^4 + 5b^4 y^4 a^2 - b^2 x^6 a^2 - 3b^6 x^2 a^2 - 3b^4 a^2 x^4 - 4a^4 x^2 b^4 \\
& + r^2 a^8 - a^2 b^8 - 2y^2 x^2 b^6 - 3b^4 x^2 y^4 - b^4 x^4 y^2 + 2a^6 x^2 y^2 - 3b^4 x^2 a^2 y^2 \\
& + 3a^4 b^2 x^2 y^2 - 2b^4 y^6 + 3b^6 r^4 + 8a^4 x^2 b^2 r^2 - y^2 b^8 - 2a^6 x^2 b^2 - 5a^4 x^4 b^2 \\
& - 3a^6 x^4 - 2b^6 r^2 a^2 + 3b^4 r^4 a^2 + 2r^6 a^4 + 4r^6 b^2 a^2 - 3r^4 a^6 - 6r^2 a^4 b^4 \\
& + 10b^2 r^2 a^2 x^2 y^2 + 6a^4 x^2 r^2 y^2 + 6a^6 x^2 r^2 + 6b^2 r^2 a^2 x^4 - 6a^4 x^2 r^4 \\
& + 6b^2 y^4 a^2 r^2 - 9b^2 y^2 r^4 a^2 - 3y^2 a^4 r^4 - 9b^2 r^4 a^2 x^2 - b^2 x^4 a^2 y^2 - b^2 x^2 y^4 a^2 \\
& - y^4 a^4 x^2 - 3a^4 x^4 y^2 + 6a^4 x^4 r^2 - 2y^2 a^6 r^2 + y^4 a^4 r^2 - b^2 y^6 a^2 - 8b^4 y^2 a^2 r^2 \\
& - 4b^2 y^2 a^4 r^2 - 2b^2 a^6 r^2 + 4b^4 x^2 a^2 r^2 - 3b^2 a^4 r^4 - 2a^4 x^6
\end{aligned} \tag{63}$$

Continuing our approach from Section 3, in order to find the solutions to the vector equation  $\nabla g = 0$ , we need to solve a system of six polynomial equations  $h_i = 0, i = 1, \dots, 5$  and  $g = 0$  in the variables  $x, y, a, b, r$  under the assumption that  $a > b > r > 0$ . Let  $I_S = \langle \nabla g, g \rangle = \langle h_1, h_2, h_3, h_4, h_5, g \rangle$  and let  $F$  be a list containing the six polynomial equations:

$$F = [h_1 = 0, h_2 = 0, h_3 = 0, h_4 = 0, h_5 = 0, g = 0] \tag{64}$$

In order to solve system (64), we can again either employ the Gröbner basis approach similar to the one shown in Appendix C in case of the ellipse, or simply use `solve` command in `Maple`. Thus we find that the affine variety  $\mathbf{V}(I_S)$  consists of the following eight singular points of which six are real and two are complex:

$$S_1 = \left( 0, \frac{\sqrt{(a^2 + b^2)(r^2 + b^2)}}{b} \right), \quad S_2 = \left( 0, -\frac{\sqrt{(a^2 + b^2)(r^2 + b^2)}}{b} \right), \tag{65}$$

$$S_3 = \left( \frac{\sqrt{(a^2 + b^2)(-a^2 + r^2)}}{a}, 0 \right), \quad S_4 = \left( -\frac{\sqrt{(a^2 + b^2)(-a^2 + r^2)}}{a}, 0 \right), \quad (66)$$

$$S_5 = \left( \sqrt{-\frac{\delta}{(a^2 + b^2)(rb)^{\frac{2}{3}}}}, -\frac{\sqrt{(rb)^{\frac{1}{3}}(a^2 + b^2)\varepsilon}}{(rb)^{\frac{1}{3}}(a^2 + b^2)} \right), \quad (67)$$

$$S_6 = \left( -\sqrt{-\frac{\delta}{(a^2 + b^2)(rb)^{\frac{2}{3}}}}, -\frac{\sqrt{(rb)^{\frac{1}{3}}(a^2 + b^2)\varepsilon}}{(rb)^{\frac{1}{3}}(a^2 + b^2)} \right), \quad (68)$$

$$S_7 = \left( \sqrt{-\frac{\delta}{(a^2 + b^2)(rb)^{\frac{2}{3}}}}, \frac{\sqrt{(rb)^{\frac{1}{3}}(a^2 + b^2)\varepsilon}}{(rb)^{\frac{1}{3}}(a^2 + b^2)} \right), \quad (69)$$

$$S_8 = \left( -\sqrt{-\frac{\delta}{(a^2 + b^2)(rb)^{\frac{2}{3}}}}, \frac{\sqrt{(rb)^{\frac{1}{3}}(a^2 + b^2)\varepsilon}}{(rb)^{\frac{1}{3}}(a^2 + b^2)} \right), \quad (70)$$

where

$$\delta = 3b^2 a^{\frac{4}{3}} r^2 - r^2 a^2 (rb)^{\frac{2}{3}} - 3a^{\frac{2}{3}} (rb)^{(1/3)} b^3 r + b^4 (rb)^{\frac{2}{3}}, \quad (71)$$

$$\varepsilon = a^4 (rb)^{\frac{1}{3}} + b^2 r^2 (rb)^{\frac{1}{3}} + 3b a^{\frac{4}{3}} r (rb)^{\frac{2}{3}} + 3a^{\frac{8}{3}} b r. \quad (72)$$

Notice that  $\delta$  becomes zero in three cases:

1. When  $r = 0$  – this is not possible since we expect  $r > 0$ ,
2. When  $r = -\frac{b^2}{a}$  – this is not possible as  $r > 0$ ,
3. When  $r = \frac{b^2}{a}$ .

Only this last value is possible thus we set  $r_{crit} = \frac{b^2}{a}$ . When  $r = r_{crit}$ , then under the assumptions that  $a > b > r > 0$ , we essentially obtain two real singular points:

$$S_1 = S_7 = S_8 = \left( 0, \frac{a^2 + b^2}{a} \right) = \left( 0, \frac{c^2}{a} \right) \quad \text{and} \quad S_2 = S_5 = S_6 = \left( 0, -\frac{a^2 + b^2}{a} \right) = \left( 0, -\frac{c^2}{a} \right), \quad (73)$$

while the complex points  $S_3$  and  $S_4$  become:

$$S_3 = \left( \frac{a^2 + b^2}{a^2} \sqrt{a^2 - b^2} i, 0 \right) \quad \text{and} \quad S_4 = \left( -\frac{a^2 + b^2}{a^2} \sqrt{a^2 - b^2} i, 0 \right). \quad (74)$$

For a hyperbola with  $a > b$ , the ratio  $l = \frac{b^2}{a}$  is called a *semi-latus rectum*. It is the distance between a focus and the hyperbola itself measured along a line perpendicular to the major axis. The value of  $l$  is also the reciprocal of the maximum curvature  $\kappa_{max}$  of the hyperbola, or, it is equal to the radius of the smallest osculating circle. [14]

To summarize, the variety  $\mathbf{V}(I_S)$  contains the following real singular points:

1. Two virtual points  $S_1$  and  $S_2$  shown in (65) when  $0 < r < r_{crit}$ . In particular, when  $a = 1, b = 1$ , and  $r = \frac{1}{2}$ , upon substituting these values into (58) we obtain the parallel lines (52) derived in Example 7. Furthermore, when these values are substituted into (65), then we obtain the two virtual singular points  $S_1$  and  $S_2$  found in the same example. The remaining six singular points are complex.
2. Two points  $S_1$  and  $S_2$  shown in (65) when  $r = r_{crit}$ . In particular, we can recover results of Example 8 when we substitute values  $a = \frac{3}{2}, b = 1$ , and  $r = \frac{2}{3}$  into the general equation (58) and the singular points (73). To be precise, three of the singular points give  $S_1$ , another three give  $S_2$  whereas the remaining two gave two distinct complex points.
3. Six points  $S_1, S_2, S_5, S_6, S_7, S_8$  shown in (65), (67), (68), (69), and (70) when  $r > r_{crit}$ . In particular we can recover in the same manner six points and the equation for the parallel lines found in Example 9 when we set  $a = \frac{3}{2}, b = 1$ , and  $r = \frac{4}{3}$ . These six singular points are displayed in Figure 9. The remaining two points are complex and distinct.

## 5 Applications and Conclusions

There are several immediate applications for the results obtained in the previous sections as well as for the general method of solving systems of polynomial and rational equations using the Gröbner basis technique. In particular, we refer to the explicit form of the polynomial  $g$  that defines the parallel lines to the conics given in (12), (32), and (58). While finding  $g$  amounts to computing the reduced Gröbner bases for the appropriate ideals and then applying the elimination theory – both being standard techniques of algebraic geometry [4], [5], [6], [8], [12] – these techniques are not used and probably not even known in the area of engineering applications where numeric, CAD generated curves, dominate. However, it does not mean that the fact that the critical value  $r_{crit}$  of  $r$  equals the reciprocal of the maximum curvature of the curve – as derived above for the non-generate conics – is not known in engineering: In fact, this fact is very well known in the theory of mechanisms when designing cams with reciprocating roller follower [11, Chapter 8], [13, Section 5.10].

In the theory of machines, the “inner” parallel line (on the concave side of the curve) is referred to as the *cam profile*, while the path followed by a center of a roller – in our case it was always given above as the polynomial  $f_1$  – is referred to as the *pitch curve* [11], [13]. The pitch curve is usually a smooth curve defined piece-wise by the pre-determined motion of the follower that could be harmonic, or determined by a polynomial of degree eight (the so called *displacement function*). This results in more complicated – than in our examples above – cam contours that may have portions which are concave, convex or flat [11, page 404]. The problem of the inner curve being “non smooth”, or, in our language, having singularity points on the smooth portion of the variety  $\mathbf{V}(g)$ , is referred to as *undercutting* – when the radius  $r$  of the roller follower is greater than  $r_{crit}$ . When  $r = r_{crit}$ , that is when the radius of the roller follower equals the minimum positive (convex) radius of the pitch curve, a cutter cutting out the cam will create a *cusp* point on the cam surface. In this latter case, while the cam contour is a continuous curve, it is not smooth at one or more points that results in a cam not running well at high speeds. In the first case when  $r > r_{crit}$ , the cutter cutting the cam out (...) *undercuts or removes material needed for cam contours in different locations and also creates a sharp point or cusp on the cam surface. This cam no longer has the same displacement function you so carefully designed* [11, page 404]. Such cams are not acceptable as the radius of the roller follower is too large. As a rule of thumb, the absolute value of the minimum radius of curvature of the cam pitch curve is preferably at least 2 or 3 time as large as the radius of the roller follower  $r$ . These situations that correspond to our cases discussed in examples 2 and 3 for the parabola, 5 and 6 for the ellipse, and 8 and 9 for the hyperbola, are depicted very well in [11, Figure 8-45] and in [13, Figure 5-29].

What differs in the approaches to the problem of parallel lines presented above and designing cam profiles is that (i) in the former approach we obtain explicit formulas for the parallel lines given by the polynomial  $g$  rather than obtain a numeric representation for it using some CAD software, (ii) in reality, cam profiles are more complicated as they are not determined by a single curve but are determined piecewise and, preferably, smooth-wise. Our approach allows for an analytic, hence exact, representation of the parallel lines, and it therefore affords an exact differential calculus to be applied to  $g$ . There is no reason why in principle this approach could not be extended to piece-wise defined pitch curves. In particular, it allows for the exact computation of the coordinates of the singular points.

In the following, we briefly show one application of the result presented in Example 4. Finally, we remark that these methods extend to non conics such as a Bézier cubic [3].

## 5.1 Parallel lines to an ellipse in the finite element method

Let us consider an ellipse determined by a polynomial  $f_1$  :

$$f_1 = 4y_0^2 + 16x_0^2 - 64 \quad (75)$$

for the values of  $a = 4$  and  $b = 2$ . Thus,  $c = 2\sqrt{3}$  and  $r_{crit} = \frac{b^2}{a} = 1$ . We will generate a finite element mesh by choosing these values for the offset  $r = 0.2, 0.4, 0.6$ . These numbers determine thicknesses of layers, or, the distances of the parallel lines to the ellipse. Number of rows in the mesh matrix is twice the number of values of  $r$  values plus 1.

**Remark 1** *The reason we are interested in generating several smooth parallel lines to the ellipse is that we want to apply the finite element analysis to composite materials that are built precisely from smooth layers of various materials. As long as the parallel lines are smooth, and this is guaranteed in the range  $0 < r < r_{crit}$ , the layers are smooth and touch at each point. This guarantees, in turn, an optimal performance of the composite.*

For each of the chosen values of  $r$ , we can now compute the corresponding polynomial  $g$  by substituting values of  $a, b, r$  into (32). Thus, we get:

$$\begin{aligned} g_1 = & -0.1083369141 \cdot 10^8 y^2 - 0.1281707488 \cdot 10^8 x^2 + 0.4086271181 \cdot 10^7 y^2 x^2 \\ & + 33792 y^4 x^4 + 10240 x^2 y^6 + 40960 x^6 y^2 - 495206.40 x^4 y^2 - 369868.80 x^2 y^4 \\ & + 259850.24 x^6 - 57180.16 y^6 + 0.1185870643 \cdot 10^7 y^4 - 564800.7168 x^4 + 1024 y^8 \\ & + 16384 x^8 + 0.3681277971 \cdot 10^8 \quad \text{when } r = 0.2, \end{aligned} \quad (76)$$

$$\begin{aligned} g_2 = & -0.1031174671 \cdot 10^8 y^2 - 0.1349849501 \cdot 10^8 x^2 + 0.4156607693 \cdot 10^7 y^2 x^2 \\ & + 33792 y^4 x^4 + 10240 x^2 y^6 + 40960 x^6 y^2 - 506265.60 x^4 y^2 - 373555.20 x^2 y^4 \\ & + 252968.96 x^6 - 56688.64 y^6 + 0.1155347251 \cdot 10^7 y^4 - 684903.6288 x^4 + 1024 y^8 \\ & + 16384 x^8 + 0.3409692932 \cdot 10^8 \quad \text{when } r = 0.4, \end{aligned} \quad (77)$$

$$\begin{aligned} g_3 = & -0.9465872249 \cdot 10^7 y^2 - 0.1456495775 \cdot 10^8 x^2 + 0.4277898445 \cdot 10^7 y^2 x^2 \\ & + 33792 y^4 x^4 + 10240 x^2 y^6 + 40960 x^6 y^2 - 524697.60 x^4 y^2 - 379699.20 x^2 y^4 \\ & + 241500.16 x^6 - 55869.44 y^6 + 0.1104343859 \cdot 10^7 y^4 - 880291.0208 x^4 \\ & + 0.2986886575 \cdot 10^8 + 1024 y^8 + 16384 x^8 \quad \text{when } r = 0.6. \end{aligned} \quad (78)$$

Thus, so far we have rows in a mesh matrix. In order to create columns, we select some  $y$  values that determine the  $y$ -coordinates of points on the ellipse through which we draw perpendicular lines to the

ellipse. The intersections of these lines with the ellipse itself as well as with the parallel lines are the mesh nodes. The number of these  $y$ -values gives the number of mesh columns minus 1. The last column is obtained by including intersection points between  $g_1$ ,  $g_2$ ,  $g_3$ , and  $f_1$  with the  $y$ -axis. For the sake of this example, we select these values:  $y = 3.75, 3, 2, 1, 0, -1, -2, -3, -3.75$  (the  $y$ -values must satisfy  $-a < y < a$ ). We display the ellipse, the parallel lines, and the nodes in Figure 10.

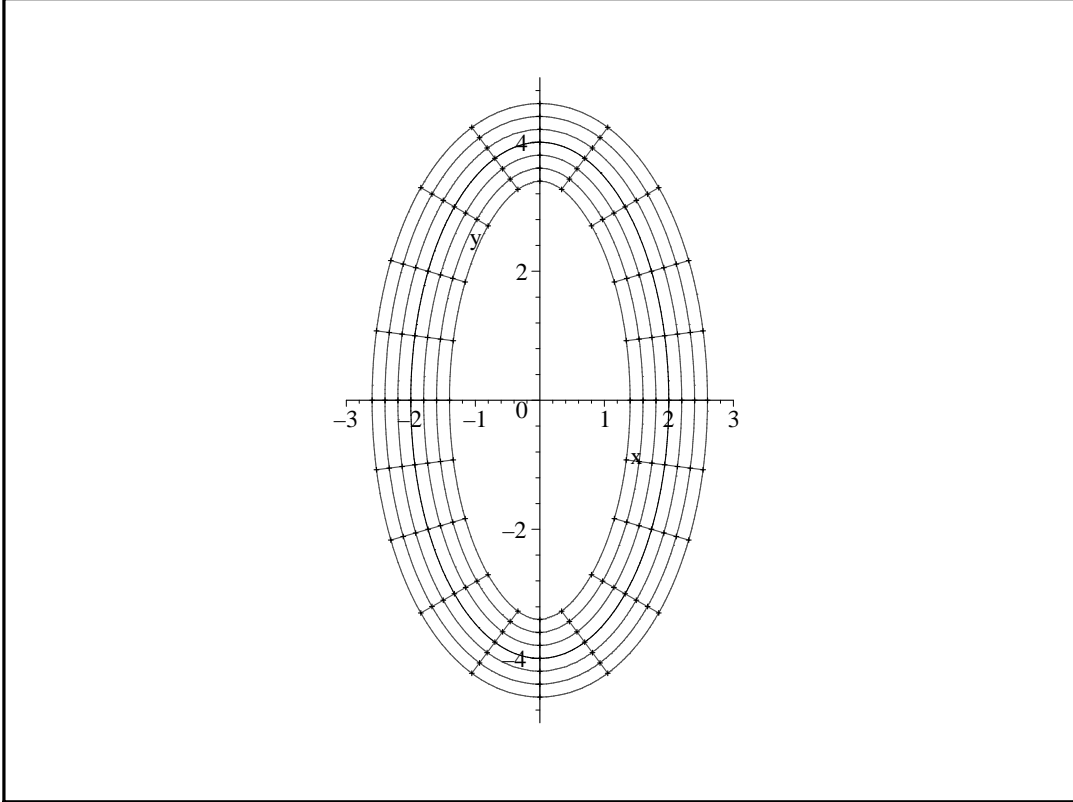


Figure 10: Ellipse with parallel lines  $g_1, g_2, g_3$  and nodes for  $a = 4$ ,  $b = 2$  and  $r = 0.2, 0.4, 0.6$

It is easy now to produce a  $7 \times 20$  mesh matrix with 140 nodes and to create a list of 120 4-node elements and a list of 30 9-node elements. Once the elements have been created in **Maple**, output has been sent to a finite element software for analysis of natural frequencies. It has been found, that there is a significant sensitivity in the higher mode of natural frequencies. For instance, when the mesh generated by exact parallel lines has been used, the higher mode of natural frequencies are much higher than the ones obtained from a mesh generated based on approximately parallel lines – several confocal ellipses.

## A Appendix: Reduced Gröbner basis for the ideal

$$I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[y_0, x_0, x, y, r, p]$$

For the ideal

$$I = \langle 4py_0 - x_0^2, y^2 - 2yy_0 + y_0^2 + x^2 - 2xx_0 + x_0^2 - r^2, 2xp - 2x_0p + x_0y - x_0y_0 \rangle \subset \mathbb{R}[y_0, x_0, x, y, r, p]$$

using lex order with  $y_0 > x_0 > x > y > r > p$ , a reduced Gröbner basis for  $I$  is given by the following fourteen polynomials:<sup>8</sup>

```
> g1:=factor(reducedG[1]);
g1 := p(-2 p r^2 y x^2 + 8 p r^2 y^3 + 8 p^2 r^2 y^2 - 32 y p^3 r^2 + 16 p^4 r^2 - 16 y^4 p^2 + 32 y^3 p^3
        - 16 p^4 y^2 + 3 r^2 x^4 + 8 p^2 r^4 + 20 p^2 r^2 x^2 - y^2 x^4 + 10 y p x^4 - x^6 - x^4 p^2 + 8 p y^3 x^2
        - 32 x^2 y^2 p^2 + 8 x^2 y p^3 - 3 r^4 x^2 + 2 r^2 x^2 y^2 + r^6 - r^4 y^2 - 8 p r^4 y)
> g2:=factor(reducedG[2]);
g2 := p(-114 x^3 p^2 y + 232 x p^3 y^2 + 40 x p^4 y - 316 x p^3 r^2 - 8 x y^3 p^2 - 2 x^3 p^3 + 16 x p^5
        - 16 p y^2 x r^2 - 12 p y^2 x_0 r^2 + 54 x y p^2 r^2 - 120 y p^2 x_0 r^2 - 32 y p^4 x_0
        + 104 p^3 x_0 r^2 - 14 p y^2 x^3 + 32 p^2 y^3 x_0 + 32 x p r^4 - 46 x^3 p r^2 - 16 p^5 x_0
        + 14 x^5 p + 27 r^4 x_0 p + 2 y x^5 + 2 y^3 x^3 - 2 y^3 x r^2 - 4 y^3 x_0 r^2 + 2 y x r^4 - 4 y x^3 r^2
        - 24 p y^4 x + 16 p y^4 x_0)
> g3:=factor(reducedG[3]);
g3 := p(4 x^2 p^3 + 56 p^3 r^2 - 4 x p^3 x_0 - 56 p^3 y^2 + 48 y^3 p^2 + 42 x^2 p^2 y - 12 x p^2 y x_0
        - 48 y p^2 r^2 - 4 x^4 p + 8 y^4 p + 14 p r^4 - 10 x^2 p r^2 - 12 x p y^2 x_0 - 8 x^2 p y^2
        + 27 x p x_0 r^2 - 22 y^2 p r^2 + 2 r^4 y + 2 y^3 x^2 + 2 y x^4 - 4 r^2 y x^2 - 4 x y^3 x_0 - 2 r^2 y^3)
> g4:=factor(reducedG[4]);
g4 := p(112 x p^4 - 112 x_0 p^4 - 120 y x p^3 + 176 y x_0 p^3 - 124 x p^2 r^2 - 80 y^2 x_0 p^2
        - 16 x_0 p^2 r^2 + 160 x p^2 y^2 - 96 x^2 p^2 x_0 + 82 x^3 p^2 - 28 x^3 y p + 4 y x_0 p r^2
        - 24 p x y^3 + 16 p y^3 x_0 + 22 x y p r^2 - 4 r^2 y^2 x_0 + 2 y^2 x^3 - 4 r^2 x^3 - 2 r^2 x y^2
        + 2 r^4 x + 3 r^4 x_0 - 3 x^2 x_0 r^2 + 2 x^5)
> g5:=factor(reducedG[5]);
g5 := 8 x p y^2 - 6 x^3 p + 7 x_0 p x^2 + 6 x p r^2 + 4 p x_0 y^2 + y x_0 x^2 - y x_0 r^2 + 8 y x p^2
        - 12 y x_0 p^2 + 2 x_0 p r^2 - 8 x p^3 + 8 x_0 p^3
> g6:=factor(reducedG[6]);
g6 := p(8 p^2 r^2 - 8 y^2 p^2 - 4 x_0 x p^2 + 4 x^2 p^2 + 4 x y x_0 p + 8 p y^3 + 2 y p x^2 - 8 y p r^2
        + 3 x_0 x r^2 - 2 r^2 y^2 + 2 x^4 + 2 r^4 + 2 x^2 y^2 - 4 x^2 r^2 - 3 x_0 x^3 - 4 x y^2 x_0)
> g7:=factor(reducedG[7]);
g7 := x_0 x^4 - 2 x^2 x_0 r^2 + 44 x^2 p^2 x_0 + 32 y^2 x_0 p^2 - 12 y x_0 p r^2 - 80 y x_0 p^3 + r^4 x_0
        + 16 x_0 p^2 r^2 + 48 x_0 p^4 - 2 x^3 y p - 38 x^3 p^2 - 40 x p^2 y^2 + 2 x y p r^2 + 56 y x p^3
        + 20 x p^2 r^2 - 48 x p^4
> g8:=factor(reducedG[8]);
g8 := 3 x_0^2 r^2 - 12 x_0^2 p^2 + 16 x y x_0 p + 32 x_0 x p^2 - 20 x^2 p^2 - 48 y^2 p^2 + 48 p^2 r^2
        - 2 x_0 x^3 + 2 x_0 x r^2 + 4 y p x^2
> g9:=factor(reducedG[9]);
g9 := -4 p y^2 - 4 p x^2 + 6 p x x_0 - 2 p x_0^2 + 4 p r^2 + x_0^2 y
```

---

<sup>8</sup>We display these polynomials in a form returned by **Maple**. Polynomials  $g_1, g_2, g_3, g_4, g_6$  can be further simplified by dividing them by  $p$  under the assumption that  $p \neq 0$ .

> `g10:=factor(reducedG[10]);`

$$g10 := -2 x0 x^2 + 3 x0^2 x + 2 x0 r^2 + 4 y x p - 8 p x0 y - 8 x p^2 + 8 x0 p^2$$

> `g11:=factor(reducedG[11]);`

$$g11 := x0^3 - 8 x p^2 + 8 x0 p^2 - 4 p x0 y$$

> `g12:=factor(reducedG[12]);`

$$g12 := 4 p y0 - x0^2$$

> `g13:=factor(reducedG[13]);`

$$g13 := -2 x p + 2 x0 p - x0 y + x0 y0$$

> `g14:=factor(reducedG[14]);`

$$g14 := y^2 - 2 y y0 + y0^2 + x^2 - 2 x x0 + x0^2 - r^2$$

## B Appendix: Reduced Gröbner basis for the ideal

$$I = \langle f_1, f_2, f_3 \rangle \subset \mathbb{R}[y_0, x_0, x, y, a, b, r]$$

For the ideal

$$I = \langle b^2 y_0^2 + a^2 x_0^2 - a^2 b^2, y^2 - 2 y y_0 + y_0^2 + x^2 - 2 x x_0 + x_0^2 - r^2, b^2 y_0 x - b^2 y_0 x_0 - a^2 x_0 y + a^2 x_0 y_0 \rangle \subset \mathbb{R}[y_0, x_0, x, y, a, b, r]$$

using lex order with  $y_0 > x_0 > x > y > a > b > r$ , a reduced Gröbner basis for  $I$  is given by the following fifteen polynomials:<sup>9</sup>

> `g1:=factor(reducedG[1]);`

---

<sup>9</sup>We display these polynomials in a form returned by `Maple`. All polynomials except  $g_{13}$  and  $g_{14}$  can be further simplified by dividing them by  $a^2, b^2$  under the assumption that  $a, b \neq 0$ .



$$\begin{aligned}
g1 := & a^2 b^2 (6 b^4 x^4 a^4 - 4 y^6 b^4 a^2 - 6 a^6 x^4 b^2 - 4 b^6 x^2 a^4 + 6 b^4 x^2 a^6 + a^4 b^8 - 2 y^2 b^8 a^2 \\
& + 6 y^2 b^6 x^2 a^2 + 6 a^6 y^2 x^2 b^2 - 10 b^4 x^2 a^4 y^2 + y^8 b^4 + b^8 r^4 + b^4 r^8 - 2 b^6 r^6 \\
& - 2 b^4 y^2 x^4 r^2 - 6 x^2 y^4 b^4 r^2 + 6 x^2 b^4 r^4 y^2 - 4 b^4 y^6 r^2 + 2 b^4 y^6 x^2 + y^4 b^4 x^4 \\
& + b^4 x^4 r^4 - 2 b^4 r^6 x^2 - 2 b^6 x^2 r^4 + 6 y^4 b^4 r^4 - 2 b^8 y^2 r^2 + 6 b^6 r^4 y^2 - 4 b^4 r^6 y^2 \\
& + 6 a^4 y^2 b^6 + 6 a^6 y^2 b^2 r^2 + 2 a^4 b^2 x^4 y^2 - 4 a^4 x^6 b^2 + 4 a^6 x^2 b^2 r^2 - 8 b^4 x^2 a^4 r^2 \\
& + 10 a^4 x^4 b^2 r^2 - 6 b^4 r^2 a^2 x^2 y^2 - 6 a^4 x^2 b^2 r^2 y^2 - 8 a^4 x^2 b^2 r^4 - 6 b^4 r^2 a^2 x^4 \\
& + 4 b^4 r^4 a^2 x^2 - 6 y^4 a^4 b^2 x^2 - 6 b^4 x^4 a^2 y^2 + 6 b^6 x^2 r^2 a^2 + 2 b^4 x^2 y^4 a^2 \\
& + 10 b^4 y^4 a^2 r^2 + 4 b^6 y^2 r^2 a^2 - 8 b^4 y^2 r^4 a^2 - 8 b^4 y^2 a^4 r^2 + 2 r^6 a^2 b^4 + a^8 b^4 \\
& - 6 a^4 y^4 r^2 b^2 + 4 a^4 y^2 r^4 b^2 - 2 a^8 b^2 x^2 + x^4 a^4 y^4 + 4 x^2 b^6 y^2 r^2 - 6 y^4 b^6 a^2 \\
& - 4 a^6 y^2 b^4 - 2 y^4 a^4 x^2 r^2 + 2 b^2 r^2 a^2 x^4 y^2 - 2 r^8 a^2 b^2 + r^8 a^4 + a^8 x^4 + a^4 x^8 \\
& - 4 a^4 x^6 r^2 + 6 a^4 x^4 r^4 + 2 b^2 r^2 a^2 x^6 - 6 b^2 r^4 a^2 x^4 - 2 a^6 r^6 + 2 a^6 x^6 + a^8 r^4 \\
& + 2 y^6 b^2 a^2 x^2 - 2 r^6 a^4 y^2 + 6 r^6 b^2 y^2 a^2 - 2 a^8 x^2 r^2 + y^4 a^4 r^4 - 6 y^4 b^2 r^4 a^2 \\
& - 2 a^6 y^2 r^4 + 4 x^2 a^6 y^2 r^2 + 6 x^2 a^4 y^2 r^4 + 2 y^4 b^2 x^2 a^2 r^2 - 10 x^2 b^2 y^2 r^4 a^2 \\
& + 2 y^6 b^2 a^2 r^2 + 6 r^6 b^2 x^2 a^2 + 4 b^2 x^4 y^4 a^2 + 2 a^4 x^6 y^2 - 6 a^6 x^4 r^2 - 4 r^6 a^4 x^2 \\
& + 2 b^2 x^6 a^2 y^2 + 6 r^4 a^6 x^2 - 6 a^4 x^4 r^2 y^2 + 2 b^6 y^6 - 2 a^8 b^2 r^2 - 2 x^2 y^4 b^6 \\
& + 6 a^4 y^4 b^4 + y^4 b^8 - 6 y^4 b^6 r^2 - 2 x^4 a^6 y^2 - 2 a^6 b^6 + 2 b^6 r^4 a^2 - 2 b^8 r^2 a^2 \\
& + 2 b^4 a^6 r^2 - 6 b^4 a^4 r^4 + 2 r^2 a^4 b^6 + 2 r^4 a^6 b^2 + 2 r^6 a^4 b^2)
\end{aligned}$$

> g2:=factor(reducedG[2]);

$$\begin{aligned}
g2 := & b^2 (-2 y a^2 b^2 + b^2 y 0 y^2 + y 0 a^2 b^2 - b^2 y 0 r^2 + 3 a^2 y x x 0 - a^2 y 0 x^2 + a^2 y^2 y 0 \\
& - a^4 y 0 + a^2 y 0 r^2 + a^4 y - a^2 y^3 - a^2 y x^2 + a^2 y r^2)
\end{aligned}$$

> g3:=factor(reducedG[3]);

$$\begin{aligned}
g3 := & a^2 (-x 0 b^2 y^2 + x 0 a^2 x^2 + x 0 a^2 b^2 - x 0 a^2 r^2 + 3 y b^2 y 0 x - 2 x a^2 b^2 + b^2 x^2 x 0 \\
& - b^4 x 0 + x 0 b^2 r^2 + b^4 x - b^2 x y^2 - b^2 x^3 + x b^2 r^2)
\end{aligned}$$

> g4:=factor(reducedG[4]);

$$\begin{aligned}
g4 := & b^2 (3 x 0 a^2 x^3 + 6 a^4 x x 0 - 3 b^2 x a^2 x 0 - 3 x x 0 a^2 r^2 + 4 a^2 y y 0 x^2 + 2 a^2 y^3 y 0 \\
& - 2 a^4 y y 0 - b^2 y 0 y^3 + 5 y y 0 a^2 b^2 - 4 a^2 y y 0 r^2 + y b^2 y 0 r^2 - 3 x^4 a^2 - 5 x^2 y^2 a^2 \\
& - 2 y^4 a^2 - 3 a^4 x^2 + 2 a^4 y^2 + 3 b^2 x^2 a^2 - y^2 a^2 b^2 - 3 a^4 b^2 + 6 x^2 a^2 r^2 + 5 y^2 a^2 r^2 \\
& + 3 a^4 r^2 + 3 b^2 a^2 r^2 - 3 r^4 a^2)
\end{aligned}$$

> g5:=factor(reducedG[5]);

$$\begin{aligned}
g5 := & b^2 (-b^4 x y 0 r^2 + a^2 b^4 y 0 x + 3 x 0 b^4 y a^2 - 5 b^4 x y a^2 + b^4 y^2 y 0 x + 2 b^2 x^3 a^2 y \\
& - 7 b^2 x a^2 y^2 y 0 - 3 x 0 b^2 y a^2 r^2 - a^2 b^2 y 0 x^3 - 3 x 0 a^4 y b^2 - 2 b^2 x a^2 y r^2 \\
& + 2 b^2 x a^2 y^3 + 5 x a^4 y b^2 + 3 x 0 b^2 y^3 a^2 + a^4 x y^2 y 0 - x^3 y a^4 + a^4 x y 0 r^2 \\
& + 3 a^4 x 0 y r^2 - a^6 x y 0 - a^4 x y^3 - a^4 y 0 x^3 + y a^4 x r^2 + a^6 x y)
\end{aligned}$$

> g6:=factor(reducedG[6]);

$$\begin{aligned}
g6 := & b^2 (-2 a^4 y^4 - 6 b^2 r^2 a^2 x x 0 + b^4 y^2 a^2 + 4 a^4 y y 0 b^2 + 4 y 0 b^4 y a^2 - b^2 x^2 a^2 y^2 \\
& + 3 b^2 x^2 a^2 r^2 + 4 b^2 y^2 a^2 r^2 - 4 a^2 y b^2 y 0 x^2 - 4 a^2 y b^2 y 0 r^2 - 2 b^4 y 0 y^3 \\
& + 3 b^4 r^2 a^2 - 3 b^2 r^4 a^2 - 3 a^4 b^4 + 2 y b^4 y 0 r^2 - b^2 y^4 a^2 + 6 a^4 b^2 x^2 + 6 a^4 b^2 r^2 \\
& - 3 a^6 b^2 + 4 a^4 y y 0 x^2 - 4 a^4 y y 0 r^2 + 3 a^6 r^2 - 3 a^4 r^4 + 2 a^4 y^3 y 0 - 2 a^6 y y 0 \\
& + 6 a^4 x^2 r^2 + 5 a^4 y^2 r^2 + 6 a^6 x x 0 - 3 x^4 a^4 - 3 x^2 a^6 - 5 a^4 x^2 y^2 + 2 a^6 y^2)
\end{aligned}$$

```

> g7:=factor(reducedG[7]);
g7 := b^2(-21 a^4 x b^4 - 6 x^3 b^4 a^2 + 2 a^6 y^2 x - 3 x^5 a^4 + 6 b^6 x a^2 + 9 a^6 b^2 x + 6 a^4 y^2 b^2 x
- 5 b^4 y^2 x a^2 - 5 y^2 a^4 x^3 - 2 y^4 a^4 x + 12 a^4 b^2 x^3 - 3 a^6 x^3 - 6 a^6 b^2 x 0
- 6 b^6 x 0 a^2 + 4 y a^4 y 0 x^3 + 2 y^3 a^4 x y 0 - 2 y a^6 x y 0 - 14 y a^4 b^2 y 0 x
- 4 y a^4 x y 0 r^2 + 5 y^2 a^4 x r^2 - 4 y 0 b^2 x^3 a^2 y + 22 y 0 b^4 x y a^2 + 6 a^4 x^3 r^2
+ 3 a^6 x r^2 - 2 b^4 x y 0 y^3 - 3 a^4 x r^4 - b^2 x^3 a^2 y^2 - b^2 y^4 x a^2 - 6 b^4 y^2 a^2 x 0
- 12 b^2 a^4 x 0 r^2 + 6 b^4 x^2 a^2 x 0 + 6 x 0 b^4 r^2 a^2 + 6 x 0 a^6 r^2 + 6 x 0 a^4 b^2 y^2
+ 2 b^4 x y y 0 r^2 + 12 x 0 a^4 b^4 + 3 b^2 x^3 a^2 r^2 - 4 y a^2 b^2 x y 0 r^2 - 3 b^2 x r^4 a^2
- 6 b^2 x^2 r^2 a^2 x 0 + 4 b^2 y^2 x a^2 r^2 + 9 b^4 r^2 a^2 x)

> g8:=factor(reducedG[8]);
g8 := b^2(4 a^4 y^3 - 2 a^4 x^2 y - 4 a^2 y^3 x^2 - 2 y^5 a^2 - 2 a^6 y - 2 a^2 b^2 y 0 x^2 + 4 a^2 b^2 y 0 y^2
+ 4 a^2 b^2 y 0 r^2 + 2 a^4 y b^2 - 2 b^2 y^3 a^2 - 2 b^4 y a^2 - b^4 y 0 r^2 + b^4 y 0 a^2 + b^4 y 0 y^2
- 3 a^4 y 0 b^2 + 2 a^4 y r^2 + 2 b^2 y a^2 r^2 + 4 b^2 y a^2 x^2 + 2 a^6 y 0 + 4 x^2 a^2 y r^2
+ 3 x^2 a^2 y^2 y 0 - 2 x^2 a^2 y 0 r^2 - 3 a^2 y^2 y 0 r^2 - b^2 y^2 y 0 x^2 + 2 b^2 y^2 y 0 r^2
+ b^2 x^2 y 0 r^2 + 4 y^3 a^2 r^2 - 2 y r^4 a^2 + r^4 a^2 y 0 - 3 a^4 y 0 r^2 - b^2 y^4 y 0 - b^2 r^4 y 0
+ 2 a^2 y^4 y 0 + a^2 y 0 x^4 + 3 x^2 a^4 y 0 - 4 a^4 y^2 y 0 - 2 x^4 y a^2)

> g9:=factor(reducedG[9]);
g9 := b^2(-4 b^2 x^2 a^2 y^2 y 0 - 2 x^2 a^2 b^2 y 0 r^2 - 2 b^4 y^4 y 0 + b^4 y^3 a^2 - b^2 y^5 a^2 - 3 y a^4 b^4
+ 4 b^4 y^2 y 0 a^2 + 4 b^4 y^2 y 0 r^2 - b^2 x^2 a^2 y^3 + 6 x^2 a^4 y b^2 - 2 a^2 y^2 b^2 y 0 r^2
+ 2 a^4 y^2 y 0 b^2 + 2 b^2 y^3 a^2 r^2 - y b^4 r^2 a^2 - y b^2 r^4 a^2 + 8 y a^4 b^2 r^2 + b^2 x^2 a^2 y r^2
+ 2 r^4 a^2 b^2 y 0 + 2 a^2 b^4 y 0 r^2 - 2 a^6 y 0 b^2 + a^6 y b^2 - 2 r^4 b^4 y 0 - 3 a^4 x^4 y
+ 4 a^6 y^3 - 2 a^4 y^5 - 5 x^2 a^4 y^3 - 2 a^6 r^2 y 0 + a^6 r^2 y + 5 a^4 r^2 y^3 - 3 a^4 r^4 y
+ 2 a^4 y^4 y 0 - 4 a^6 y^2 y 0 + 2 a^6 x^2 y 0 + 2 a^8 y 0 + 6 a^4 x^2 y r^2 + 4 a^4 x^2 y^2 y 0
- 4 a^4 r^2 y 0 y^2 - a^6 x^2 y - 2 a^8 y)

> g10:=factor(reducedG[10]);
g10 := b^2(2 a^6 b^4 - 2 a^4 b^6 - 16 a^4 y^4 b^2 + 2 a^4 x^6 - 4 b^4 r^4 a^2 + 2 b^6 r^2 a^2 - 4 b^6 y 0 y^3
+ 3 a^4 x^4 y^2 - 2 a^8 y^2 + 4 a^6 y^4 - 4 a^6 r^2 b^2 + 4 a^4 b^2 x^2 y^2 + 9 a^6 y^2 b^2 + 2 b^4 a^4 r^2
+ 6 b^4 x^2 a^4 - 6 y^2 a^4 x^2 r^2 + 2 r^6 b^2 a^2 - 4 x^2 a^6 r^2 + 6 x^2 a^4 r^4 + 6 b^2 x^4 y^2 a^2
- 6 a^4 x^4 r^2 + 3 y^2 a^2 r^4 b^2 + 13 y^4 b^2 a^2 x^2 - 12 y^4 b^2 a^2 r^2 - 2 r^6 a^4 + 2 r^4 a^6
+ a^4 y^4 r^2 + 3 a^4 y^2 r^4 + 5 a^6 y^2 r^2 + 2 a^6 x^4 + 2 b^2 r^2 a^2 x^4 - 4 x^2 b^2 r^4 a^2
- 9 y^2 b^2 r^2 a^2 x^2 + 7 y^6 a^2 b^2 - 6 x^4 a^4 b^2 - 4 b^4 x^2 r^2 a^2 + 2 y^5 a^4 y 0 - 4 y^3 a^6 y 0
+ 2 y a^8 y 0 + 6 b^4 y^5 y 0 + 6 b^2 r^2 a^2 x^2 y y 0 - 4 b^2 x^2 a^2 y^3 y 0 + 14 b^2 y^3 a^2 y 0 r^2
- 6 b^2 a^2 y r^4 y 0 - 4 b^4 x^2 y y 0 r^2 + 4 b^4 x^2 y^3 y 0 - 8 b^2 y^5 a^2 y 0 + 6 a^6 x^2 y y 0
- 6 a^6 y r^2 y 0 - 12 b^4 y^3 y 0 r^2 + 6 b^4 y r^4 y 0 + 2 a^4 b^2 r^4 - 2 y^6 a^4 + 18 a^4 y^3 b^2 y 0
- 10 a^6 y y 0 b^2 + 8 a^4 y 0 b^4 y + b^4 y^2 a^2 r^2 + 4 b^6 y y 0 r^2 - 8 a^4 y b^2 y 0 x^2
+ 28 a^4 y b^2 y 0 r^2 - 26 a^2 y b^4 y 0 r^2 + 4 a^4 b^2 x^2 r^2 - 18 a^4 b^2 y^2 r^2 - 12 a^2 b^4 y 0 y^3
- x^2 y^4 a^4 - 4 x^2 a^6 b^2 + 3 y^4 b^4 a^2 - 7 a^4 b^4 y^2 + 6 y^2 b^6 a^2 - 12 b^4 x^2 y^2 a^2
- 5 a^6 y^2 x^2)

> g11:=factor(reducedG[11]);

```

```

g11 := b^2(-12 b^4 x^4 a^4 + 18 y^6 b^4 a^2 + 14 b^6 x^2 a^4 - 6 b^4 x^2 a^6 - 5 a^4 b^8 + 15 y^2 b^8 a^2
- 27 y^2 b^6 x^2 a^2 + 6 a^6 y^4 b^2 - 8 a^6 y^2 x^2 b^2 - 6 b^4 x^2 a^4 y^2 - 11 a^4 y^2 b^6
+ 22 a^6 y^2 b^2 r^2 + 15 a^4 b^2 x^4 y^2 + 24 b^6 y^5 y0 + 2 a^4 x^6 b^2 + 16 a^6 x^2 b^2 y y0
- 24 b^4 x^2 a^4 y y0 - 8 a^6 x^2 b^2 r^2 + 6 b^4 x^2 a^4 r^2 - 9 a^4 x^4 b^2 r^2 - 33 b^4 r^2 a^2 x^2 y^2
+ 18 b^4 r^2 a^2 x^2 y y0 - 29 a^4 x^2 b^2 r^2 y^2 - 4 a^4 x^2 b^2 r^2 y y0 + 12 a^4 x^2 b^2 r^4
+ 3 b^4 r^2 a^2 x^4 - 10 b^4 r^4 a^2 x^2 - 6 b^4 x^2 a^2 y^3 y0 + 8 x^2 b^6 y^3 y0 - 8 x^2 b^6 y y0 r^2
- 42 b^6 y^3 y0 a^2 - 48 b^6 y^3 y0 r^2 - 7 y^4 a^4 b^2 x^2 + 50 b^4 y^3 a^2 y0 r^2 + 9 b^4 x^4 a^2 y^2
- 9 b^6 x^2 r^2 a^2 + 43 b^4 x^2 y^4 a^2 - 22 b^4 y^5 a^2 y0 + 50 b^4 y^3 a^4 y0 - 29 b^4 y^4 a^2 r^2
+ 10 b^6 y^2 r^2 a^2 + 4 b^4 y^2 r^4 a^2 - 73 b^4 y^2 a^4 r^2 + 7 r^6 a^2 b^4 + a^8 b^4 + 86 y a^4 b^4 y0 r^2
- 78 y a^2 b^6 y0 r^2 - 10 y^3 b^8 y0 - 28 y a^6 b^4 y0 + 24 y a^4 b^6 y0 + 10 y b^8 y0 r^2
- 28 a^2 y r^4 b^4 y0 + 4 a^4 y^3 r^2 b^2 y0 - 18 a^6 y r^2 y0 b^2 + 4 a^4 y r^4 b^2 y0
- 9 a^4 y^4 r^2 b^2 + 14 a^4 y^2 r^4 b^2 + 24 y b^6 r^4 y0 - 2 a^8 b^2 x^2 - 6 y^3 a^6 y0 b^2
+ 4 y a^8 y0 b^2 + x^4 a^4 y^4 + y^4 b^6 a^2 - 4 a^8 y^2 b^2 + 22 a^6 y^2 b^4 - 2 y^3 b^2 r^2 a^2 y0 x^2
- 2 y^4 a^4 x^2 r^2 - 2 x^4 b^4 y y0 r^2 - b^2 r^2 a^2 x^4 y^2 - r^8 a^2 b^2 + r^8 a^4 + a^8 x^4 - 2 y^8 b^2 a^2
+ a^4 x^8 - 4 a^4 x^6 r^2 + 6 a^4 x^4 r^4 + b^2 r^2 a^2 x^6 - 3 b^2 r^4 a^2 x^4 - 2 a^6 r^6 + 2 a^6 x^6 + a^8 r^4
- y^6 b^2 a^2 x^2 + 2 x^2 y b^4 r^4 y0 - 2 r^6 a^4 y^2 + 5 r^6 b^2 y^2 a^2 - 2 a^8 x^2 r^2 + y^4 a^4 r^4
- 9 y^4 b^2 r^4 a^2 + 2 a^2 y^7 b^2 y0 - 2 a^6 y^2 r^4 - 4 y^5 b^2 a^2 y0 r^2 + 2 y^3 a^2 r^4 b^2 y0
+ 4 x^2 a^6 y^2 r^2 + 6 x^2 a^4 y^2 r^4 + 5 y^4 b^2 x^2 a^2 r^2 - 7 x^2 b^2 y^2 r^4 a^2 + 2 y^5 b^2 a^2 y0 x^2
- 4 x^2 b^4 y^3 y0 r^2 + 7 y^6 b^2 a^2 r^2 + 3 r^6 b^2 x^2 a^2 + 2 x^2 b^4 y^5 y0 + 4 b^2 x^4 y^4 a^2
+ 2 a^4 x^6 y^2 - 6 a^6 x^4 r^2 - 4 r^6 a^4 x^2 + 2 x^4 b^4 y^3 y0 + 3 b^2 x^6 a^2 y^2 + 6 r^4 a^6 x^2
- 6 a^4 x^4 r^2 y^2 - 2 a^8 b^2 r^2 - 41 a^4 y^4 b^4 - 2 x^4 a^6 y^2 + 4 a^6 b^6 - 11 b^6 r^4 a^2
+ 5 b^8 r^2 a^2 - 10 b^4 a^6 r^2 + 2 b^4 a^4 r^4 + 7 r^2 a^4 b^6 + 8 r^4 a^6 b^2 - 5 r^6 a^4 b^2)
> g12:=factor(reducedG[12]);

```

$$\begin{aligned}
g12 := & b^2(-3y^6b^4a^2x^2 - 9a^4r^6b^4 + 2a^6b^8 + 2a^8r^4b^2 - 19a^6r^2b^6 + 32a^6b^4r^4 \\
& - 13a^8b^4r^2 + 20a^4b^8r^2 - 10a^4b^6r^4 + 8a^8b^6 + 24y^2b^{10}a^2 - 15a^6r^6b^2 \\
& + 10a^6x^6b^2 - 17a^6x^4y^2b^2 - 39y^2b^8x^2a^2 + 23a^6y^2x^2b^4 - 16a^8x^2y^2b^2 \\
& - 37a^4y^4x^2b^4 + 24x^2y^4a^6b^2 - 16b^{10}y^3y0 + 2b^4x^6a^4 + 94y^4b^6x^2a^2 \\
& + 9y^2b^6x^4a^2 - 52b^6x^2a^4y^2 + 48b^4x^4a^4y^2 - 8b^{10}a^4 - 18a^6b^4x^4 + 8b^{10}r^2a^2 \\
& - 22b^8r^4a^2 + 21b^6r^6a^2 + 16b^{10}y0r^2 - 8r^8a^2b^4 + 8r^8a^4b^2 - 16b^6x^4a^4 \\
& + 21b^8x^2a^4 + 2y^3b^4r^2a^2y0x^2 - 20y^4a^4b^2x^2r^2 - 2a^{10}b^4 - 2a^{10}x^4 \\
& + 4a^{10}b^2x^2 + a^4x^{10} - 3a^8x^6 - 19a^8b^4x^2 + 14a^8x^4b^2 - 16a^4b^2y^6x^2 \\
& - 2a^6x^6y^2 + 5y^4a^4x^6 + 4a^4x^8y^2 - 6x^4a^6y^4 + 6x^4a^8y^2 + 2y^6a^4x^4 + 6b^6x^2a^6 \\
& - 14y^4b^8a^2 + 24b^6y^6a^2 + 20a^6y^2b^6 - 4x^4b^6y0r^2 - 8b^4r^2a^2x^4y^2 \\
& - 8a^4x^6b^2r^2 + 24a^4x^4b^2r^4 + b^4r^2a^2x^6 - 10b^4r^4a^2x^4 + 4x^4b^6y^3y0 \\
& + 3b^4x^6a^2y^2 + 4b^4a^4x^2r^4 + 40r^4a^6x^2b^2 - 10r^2a^4x^2b^6 + 17r^6b^4x^2a^2 \\
& - 24r^6a^4x^2b^2 - 20a^4x^4b^2r^2y^2 - 30x^2b^4y^2r^4a^2 - 2x^2a^2y0r^4b^4y0 \\
& + 40x^2a^4y^2r^4b^2 - 12b^6x^2r^4a^2 + 8x^2b^6y^5y0 + 6b^4x^4y^4a^2 + 3b^6x^4a^2r^2 \\
& - 35a^6x^4b^2r^2 + 16y^4b^4x^2a^2r^2 + 116y^3a^2b^6y0r^2 - 4y^5b^4a^2y0r^2 \\
& + 2y^3a^2r^4b^4y0 + 34x^2a^6y^2b^2r^2 - 20x^2b^6y^3y0r^2 + 10b^8x^2y0y^3 \\
& - 13b^8x^2r^2a^2 - 50y^5b^6y0a^2 - 10y^5b^6y0r^2 + 39y^6b^4a^2r^2 - 8y^4a^4b^2x^4 \\
& - 67y^4a^4b^4r^2 - 31y^4b^6a^2r^2 - 8y^3a^6b^4y0 + 8y^3b^6r^4y0 + 24y^4a^6r^2b^2 \\
& - 16y^6a^4r^2b^2 + 28y^4a^4r^4b^2 - 56y^4b^4r^4a^2 + 3b^4a^4x^4r^2 + 6b^4a^6x^2r^2 \\
& + 4a^8y^2b^4 - 10y^8b^4a^2 + 100a^4y^3b^6y0 - 17a^6y^2b^2r^4 - 16a^8y^2b^2r^2 \\
& - 178a^4b^6y^2r^2 + 48a^4b^4r^4y^2 - 94b^8y^3y0a^2 - 114b^8y^3y0r^2 + 32b^8y^2a^2r^2 \\
& - 14b^6r^4y^2a^2 - 20r^6a^4b^2y^2 + 35r^6b^4y^2a^2 + 6ya^8b^4y0 - 54ya^6b^6y0 \\
& + 56yb^8r^4y0 - 12b^4ya^4x^2y0r^2 + 30b^4ya^6x^2y0 - 2b^6y^3a^2y0x^2 \\
& - 48b^6ya^4y0x^2 + 32b^6ya^2x^2y0r^2 - 82b^4y^2a^4x^2r^2 - 10b^8x^2y0r^2 \\
& - 70b^6y^2x^2a^2r^2 + 8a^4y^3b^4y0r^2 - 34ya^6b^4y0r^2 + 170ya^4b^6y0r^2 \\
& + 10ya^4r^4b^4y0 - 66ya^2b^6r^4y0 - 152yb^8a^2y0r^2 + 77a^6y^2b^4r^2 \\
& + 58b^8y^5y0 + 48ya^4b^8y0 - 52a^4y^4b^6 + 2b^4y^9y0 + 4b^6y^7y0 + 6b^4y^7y0x^2 \\
& - 12a^6y^4b^4 + 12x^2a^6y^4r^2 + 10x^2y^3b^4r^4y0 - 12x^2a^8y^2r^2 + 4r^8a^4y^2 \\
& - 7r^8b^2y^2a^2 + 2y^6a^4r^4 - 2y^3b^4r^6y0 - 4y^6a^4x^2r^2 - 6b^4y^7y0r^2 \\
& + 11b^2y^6x^4a^2 + 4b^2y^8x^2a^2 + 4b^2y^8a^2r^2 - 13b^2y^6r^4a^2 + 6b^4y^5r^4y0 \\
& + 5r^8a^4x^2 - 4r^8b^2x^2a^2 - 16a^8b^2x^2r^2 + 9a^8x^4r^2 - 9a^8r^4x^2 - 5a^4x^8r^2 \\
& + 10a^4x^6r^4 + b^2r^2a^2x^8 - 4b^2r^4a^2x^6 + 2x^6b^4y^3y0 + 3b^2x^8a^2y^2 \\
& + 6r^6b^2x^4a^2 - 10r^6a^4x^4 + 6x^4b^4y^5y0 + 10b^2x^6y^4a^2 - 12x^4b^2y^2r^4a^2 \\
& + 24x^4a^4y^2r^4 - 5y^4b^2x^4a^2r^2 - 10x^4b^4y^3y0r^2 - 6a^6x^2y^2r^4 - 15y^4a^4x^4r^2 \\
& - 2x^6b^4y0r^2 - 2b^2r^2a^2x^6y^2 + 4r^4x^4b^4y0y0 + 15x^2y^4a^4r^4 \\
& - 20x^2y^4b^2r^4a^2 - 16x^2r^6a^4y^2 + 18x^2r^6b^2y^2a^2 + 12x^2yb^6r^4y0 \\
& + 6a^6x^4r^2y^2 - 16a^4x^6r^2y^2 - 14b^4y^5x^2y0r^2 + 2b^2y^6x^2a^2r^2 - 6y^4a^6r^4 \\
& - 5y^4r^6a^4 + 15y^4r^6b^2a^2 + 6a^8y^2r^4 + 2a^6r^6y^2 + 2yb^4r^6a^2y0 - 2yb^6r^6y0 \\
& + 20y^6a^4b^4 + 4a^{10}x^2r^2 - 2r^6x^2b^4y0y0 - 2a^{10}r^4 + 4a^{10}b^2r^2 + 3a^8r^6 \\
& + r^{10}a^2b^2 - r^{10}a^4)
\end{aligned}$$

> g13:=factor(reducedG[13]);

$$g13 := y^2 - 2y y0 + y0^2 + x^2 - 2x x0 + x0^2 - r^2$$

> g14:=factor(reducedG[14]);

$$g14 := b^2 y0 x - b^2 y0 x0 - a^2 x0 y + a^2 x0 y0$$

> g15:=factor(reducedG[15]);

$$g15 := b^2 (x0 y0 x + y y0^2 - y0 x^2 - y0 y^2 - y0 a^2 + y0 r^2 + y a^2)$$

## C Appendix: Reduced Gröbner basis for the ideal

$$I_S = \langle h_1, h_2, h_3, h_4, h_5, g \rangle \subset \mathbb{R}[x, y, a, b, r]$$

In order to solve system (38), we have found a reduced Gröbner basis for  $I_S$  using the total term order `tdeg` for  $y > x > a > b > r$  with ties broken by inverse lexicographical order. We used package `Groebner` from `Maple 8` supplemented by a small custom package `RJGrobner` written by the Authors. [2] The reduced basis contains fourteen homogeneous polynomials with degrees ranging from 10 through 15 and the number of monomials terms ranging from 51 through 112. We will not display these polynomials here. However, here is the factorization the fourth polynomial  $b_4$  in the basis:

$$\begin{aligned} b_4 = & b^2 x (4y^2 r^4 a^2 - 5y^4 a^2 r^2 + a^4 r^2 y^2 - 2b^2 x^2 r^4 + b^2 x^4 r^2 - 9x^2 b^2 y^2 a^2 - 11b^2 y^2 a^2 r^2 \\ & - 2x^2 b^4 r^2 + 4a^6 r^2 - 2a^8 - b^6 a^2 - a^4 b^4 + 4a^6 b^2 + 6y^2 a^6 - a^4 r^4 - r^6 a^2 \\ & - 4y^4 b^4 - 6y^4 a^4 - 4a^6 x^2 + 2y^6 a^2 - a^4 x^4 - b^6 y^2 + b^6 r^2 + r^6 b^2 - 4b^4 r^4 \\ & + 2a^4 b^2 x^2 - 9a^4 b^2 y^2 + 2x^2 b^4 y^2 + 5b^4 y^2 a^2 - a^4 x^2 y^2 - 3b^2 a^2 x^4 + 6b^2 y^4 a^2 \\ & + 3x^2 b^4 a^2 - y^6 b^2 + a^2 x^6 - 2b^2 x^2 a^2 r^2 + 2a^4 x^2 r^2 - 3a^2 x^4 r^2 + 3a^2 x^2 r^4 \\ & - 8x^2 y^2 a^2 r^2 + 4b^2 x^2 y^2 r^2 + 8b^4 y^2 r^2 + 5b^2 r^4 a^2 + 5b^4 r^2 a^2 - 3b^2 r^4 y^2 \\ & + 3b^2 y^4 r^2 - 10b^2 a^4 r^2 - b^2 x^4 y^2 - 2y^4 b^2 x^2 + 4a^2 x^4 y^2 + 5a^2 y^4 x^2) \end{aligned} \quad (79)$$

It appears, that since  $b > 0$ , one possible solution is  $x = 0$ . Upon substituting  $x = 0$  into the list  $F$  shown in (38), one gets

$$\begin{aligned} & [0, -y A(2b^2 y^4 + b^4 y^2 - 4b^2 r^2 y^2 - 4b^2 y^2 a^2 + y^2 a^2 r^2 - b^4 a^2 - b^4 r^2 + 2b^2 r^4 + 2a^4 b^2 - a^4 r^2 - r^4 a^2), \\ & - A(-2b^2 y^4 + r^2 y^4 - b^4 y^2 + 2b^2 r^2 y^2 + 4b^2 y^2 a^2 - 3y^2 a^2 r^2 - 2r^4 y^2 + b^4 a^2 - b^4 r^2 - 2a^4 b^2 \\ & + 2b^2 a^2 r^2 + r^6 + 2a^4 r^2 - 3r^4 a^2), \\ & (y + r - a)(y + r + a)(a - y + r)(-a - y + r)(-y^2 + r^2 + a^2 - 2b^2) A, \\ & A(-y^4 a^2 + 2b^2 y^4 + b^4 y^2 - 4b^2 r^2 y^2 - 2b^2 y^2 a^2 + 3y^2 a^2 r^2 + 2a^4 y^2 + b^4 a^2 - b^4 r^2 - 2b^2 a^2 r^2 \\ & + 2b^2 r^4 + 3a^4 r^2 - 2r^4 a^2 - a^6), \\ & (y + r - a)(y + r + a)(a - y + r)(-a - y + r) A^2] \end{aligned} \quad (80)$$

where

$$A = -y^2 b^2 - b^4 - r^2 a^2 + b^2 a^2 + b^2 r^2. \quad (81)$$

It can be easily verified that setting  $y = -r - a$  or  $y = r + a$  in (80) does not yield any critical values for  $r$ , as the system becomes, respectively,

$$F_1 = [0, -2(r+a)(b^2+ra)^4ra, 2(b^2+ra)^4r(r+a), 0, -2(b^2+ra)^4a(r+a), 0], \quad (82)$$

$$F_4 = [0, 2(r+a)(b^2+ra)^4ra, 2(b^2+ra)^4r(r+a), 0, -2(b^2+ra)^4a(r+a), 0] \quad (83)$$

as  $a > b > r > 0$  and is insolvable. However, setting  $y = -r + a$  or  $y = r - a$  does yield  $r_{crit} = \frac{b^2}{a}$  and makes the system solvable in each case:

$$F_2 = [0, 2(r-a)(-b^2+ra)^4ra, 2(-b^2+ra)^4r(r-a), 0, 2(-b^2+ra)^4a(r-a), 0], \quad (84)$$

$$F_3 = [0, -2(r-a)(-b^2+ra)^4ra, 2(-b^2+ra)^4r(r-a), 0, 2(-b^2+ra)^4a(r-a), 0] \quad (85)$$

Another possibility for solving the system is to set

$$y^2 = \frac{-b^4 - r^2a^2 + b^2a^2 + b^2r^2}{b^2} \quad (86)$$

Thus,

$$\left( x = 0, y = \pm \sqrt{\frac{-b^4 - r^2a^2 + b^2a^2 + b^2r^2}{b^2}} \right) \quad (87)$$

provide two solutions to the system without any condition on  $r$  except that  $r > 0$ . These are the two virtual singular points found earlier that are always present for  $a > b > r > 0$ . Observe also that the critical value of  $r$  satisfies

$$r_{crit} = \frac{b^2}{a} = \left( \frac{b}{a} \right) b < b \quad \text{since} \quad a > b.$$

Another possibility to solve the system is to set

$$y^2 = r^2 + a^2 - 2b^2 \quad (88)$$

When we substitute the above expressions for  $y^2$  into  $F$ , we get

$$\begin{aligned} & [0, -6y(-b^2+ra)^2(b^2+ra)^2b^2, 6(-b^2+ra)^2(b^2+ra)^2(-b+r)(b+r), 0, \\ & 6(-b^2+ra)^2(b^2+ra)^2(b-a)(b+a), \\ & (a+r-y)(a+r+y)(r-a+y)(-a+r-y)(-b^2+ra)^2(b^2+ra)^2] \end{aligned} \quad (89)$$

The above shows that in this case in order to satisfy the equations of the variety of singular points  $\mathbf{V}(I_S)$  we must also have  $r = r_{crit} = \frac{b^2}{a}$ . Hence we get

$$\left( x = 0, y = \pm \left( \frac{a^2 - b^2}{a} \right) = \frac{c^2}{a} \right) \quad \text{when} \quad r = \frac{b^2}{a}, \quad (90)$$

which is the special case of the above singular points discussed above.

Finally, we have special cases that we are not interested in but we mention them for completeness. They result when we set  $y = 0$  in the list  $F$  (in addition to  $x = 0$ ). The first case is when  $a = b$ , that is, when the ellipse is a circle. The second case is when  $r = b$  or  $r = a$ .

Thus, to summarize, we have these two cases of singular points when  $x = 0$  :

1.  $\left( x = 0, y^2 = -\frac{b^2r^2+b^4+a^2r^2-a^2b^2}{b^2} \right)$  and  $r$  is arbitrary as long as  $b > r > 0$ . This yields singular points that are always present.

2.  $\left(x = 0, y = \pm(-r + a) = \pm\left(\frac{c^2}{a}\right)\right)$  when  $r = r_{crit} = \frac{b^2}{a}$  and  $c^2 = a^2 - b^2$ . This yields singular points in the special case  $r = r_{crit}$ .

In order to complete the analysis of all solutions of the system  $F$ , we need to consider the case when  $x, y \neq 0$ . The fourteen-polynomial reduced Gröbner basis found for  $I_S$  when solving system (38), could be simplified by dividing each polynomial by  $x, y, a^2$ , or  $b^2$ , as appropriate. Let  $I \subset \mathbb{R}[x, y, a, b, r]$  denote the ideal generated by these fourteen simplified polynomials. In an effort to eliminate variable  $y$  from some polynomials, we computed<sup>10</sup> the Gröbner basis for the first elimination ideal  $I_x = I \cap \mathbb{R}[x, a, b, r]$  for the lexicographic order  $y > r > a > b > x$ . The first polynomial in this new thirteen-polynomial homogeneous basis for  $I_x$  is of degree 12. It is the square of the following polynomial  $g_x \subset \mathbb{R}[x, a, b, r]$  of degree 6 in  $x$ :

$$\begin{aligned} g_x = & (b^6 - 3b^4a^2 + 3a^4b^2 - a^6)x^6 + (6b^6a^2 + 3a^6r^2 - 6b^2a^4r^2 + 3b^4r^2a^2 - 3b^8 - 3a^4b^4)x^4 \\ & + (-21b^4a^4r^2 + 3b^{10} - 3b^8a^2 + 21b^6r^2a^2 + 3a^4b^2r^4 - 3r^4a^6)x^2 \\ & - 3b^4a^4r^4 - b^{12} + 3b^8r^2a^2 + a^6r^6 \end{aligned} \quad (91)$$

Upon substituting  $z = x^2$  this becomes an easily solvable cubic equation for  $z$  that has the following solution:

$$z = \frac{3b^2a^2r^2 - r^2a^2(abr)^{\frac{2}{3}} - 3b^3(abr)^{\frac{1}{3}}ar + b^4(abr)^{\frac{2}{3}}}{(abr)^{\frac{2}{3}}(-a^2 + b^2)} = -\frac{\delta}{(abr)^{\frac{2}{3}}(-a^2 + b^2)} \quad (92)$$

where  $\delta$  is given in (45). Thus, we have derived the  $x$ -coordinates of the singular points  $S_5, S_6, S_7$  and  $S_8$  listed in (41), (42), (43), and (44).

To find the  $y$  coordinates of these four singular points, we proceed in a similar fashion except that we eliminate the  $x$  variable. Thus, compute the first elimination ideal  $I_y = I \cap \mathbb{R}[y, a, b, r]$  for the lexicographic order  $x > r > a > b > y$ . The first polynomial in this new six-polynomial homogeneous basis for  $I_y$  is of degree 12. It is the square of the following polynomial  $g_y \subset \mathbb{R}[y, a, b, r]$  of degree 6 in  $y$ :

$$\begin{aligned} g_y = & (-b^6 + 3b^4a^2 - 3a^4b^2 + a^6)y^6 + (3b^2a^4r^2 + 3b^6r^2 - 3a^4b^4 + 6a^6b^2 - 3a^8 - 6b^4r^2a^2)y^4 \\ & + (3b^4a^2r^4 - 3b^6r^4 - 21b^4a^4r^2 + 3a^{10} + 21a^6b^2r^2 - 3a^8b^2)y^2 + b^6r^6 \\ & - 3b^4a^4r^4 - a^{12} + 3a^8b^2r^2 \end{aligned} \quad (93)$$

Upon substituting  $z = y^2$  this becomes an easily solvable cubic equation for  $z$  that has the following solution:

$$z = \frac{-a^4(abr)^{\frac{2}{3}} + 3(abr)^{\frac{1}{3}}bra^3 - 3b^2a^2r^2 + b^2r^2(abr)^{\frac{2}{3}}}{(abr)^{\frac{2}{3}}(a^2 - b^2)^2} = \frac{(-a^2 + b^2)\varepsilon}{(abr)^{2/3}(-a^2 + b^2)^2} \quad (94)$$

where  $\varepsilon$  is given in (46). Thus, we have derived the  $y$ -coordinates of the singular points  $S_5, S_6, S_7$  and  $S_8$  listed in (41), (42), (43), and (44).

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<sup>10</sup>We used `gbasis` procedure from the `Groebner` package with the total order `lexdeg([y],[r,a,b,x])` indicating that the variable  $y$  needed to be eliminated.

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